CS 161: Design and Analysis of Algorithms

Greedy Algorithms 1:
Shortest Paths In Weighted Graphs

- Greedy Algorithm
- BFS as a greedy algorithm
- Shortest Paths in Weighted Graphs
- Dijsktra’s Algorithm

Greedy Algorithms

- Build up solution, making most obvious decision at each step
- Example: Making change
  - How can I make x cents using the fewest number of coins/bills?

Making Change

- General problem:
  - Given integer coin values v_1, … v_n, and a target amount W, find a set of coins (allowing repetition) whose total value is w that minimizes the total number of coins
- Greedy Algorithm:
  - If x = 0, give nothing and stop
  - Otherwise, find the largest $v_i \leq W$, add v_1 to the set, and subtract v_1 from W. Repeat

Making Change

- For U.S. currency, we have $v_1 = 1$, $v_2 = 5$, $v_3 = 10$, $v_4 = 25$ (ignore higher values)
- Does greedy algorithm give best solution?

Greedy Algorithm Optimal?

- Claim: greedy is optimal for U.S. coins
- Equivalent to showing the following: For any amount W, there is an optimal solution using the largest v_i that is at most W

Greedy Algorithm Optimal

- Proof of equivalence:
  - If claim true, then greedy is optimal, and greedy uses largest v_i that is at most W
  - Other direction: Assume true for all W’ < W. Say greedy uses g coins, optimal uses p coins
  - Let P be an optimal solution for W that uses largest v_i
  - Greedy first picks v_i, then solves W- v_i
  - By induction, greedy optimal on W- v_i, g-1 coins
  - P without v_i is a solution for W- v_i with p-1 coins
  - Therefore, $p-1 \geq g-1$, so $p \geq g$

Greedy Algorithm Optimal?

- Claim: greedy is optimal for U.S. coins
- Proof:
  - Must have at most 2 dimes (otherwise can replace 3 dimes with quarter and nickel)
  - If 2 dimes, no nickels (otherwise can replace 2 dimes and 1 nickel with a quarter)
  - At most 1 nickel (otherwise can replace 2 nickels with a dime)
  - At most 4 pennies(otherwise can replace 5 pennies with a nickel

Greedy Algorithm Optimal?

- In optimal solutions:
  - Total value of pennies: < 5 cents
  - Total value of pennies and nickels: < 10 cents
  - Total value of non-quarters: < 25 cents
- Therefore we always use the largest coin, so greedy optimal

Making Change

- Is greedy algorithm optimal in general?
  - Say we have 20 cent pieces as well
  - What if w = 40?
  - Optimal: two 20 cent pieces (2 coins)
  - Greedy: first picks quarter, then dime, then nickel (3 coins)
- In general, greedy algorithm does not give optimal solution for the Making Change Problem!

Making Change

- What properties of coin values lets greedy be optimal?
- Let r_t = Ceiling(v_{i+1}/v_i), $s_t = r_t v_t$
- Theorem: If, for each t = 1,…,n-1, greedy algorithm outputs fewer than r_t coins for value $W_t = s_t – v_{t+1}$, then greedy always optimal

Example: U.S. Currency

t	1	2	3	4
v_t	1	5	10	25
r_t	5	2	3	N/A
s_t	5	10	30	N/A
W_t	0	0	5	N/A
Greedy(W_t)	0	0	1	N/A

Example: U.S. Currency with 20 Cent Coin

t	**1**	**2**	**3**	**4**	**5**
v_t	1	5	10	20	25
r_t	5	2	2	2	N/A
s_t	5	10	20	40	N/A
W_t	0	0	0	15	N/A
Greedy(W_t)	0	0	0	2	N/A

Greedy Algorithms

- General Goal: give a simple greedy algorithm, prove that it gives optimal solution
- Proving optimality is usually the hard part

BFS as Greedy Algorithm

- Problem: Given a source node v, compute the distances to nodes in graph
- Approach: Set all distances to ∞, update greedily
  - Set distance(u) = ∞ for u ≠ v
  - Set distance(v) = 0
  - Repeatedly process nodes:
    - Find closest node u that hasn’t been processed
    - For each edge (u,w), update distance to w

BFS as Greedy Algorithm

- When at a node u, to update distance to w:
  - distance(w) = Min(distance(w),distance(u) + 1)
- Find closest unprocessed node by keeping queue
  - queue contains unprocessed nodes that have distance < ∞
  - Works because we always add nodes to queue in in order of increasing distance, distances never updated once in queue

Why Greedy?

- We update neighbors of **closest** nodes first

Why Correct?

- Let d(u) be correct distance to u
- Claim 1: All nodes have distance(u)≥d(u)

Proof

- True at beginning. Inductively assume true for first i-1 updates
- Let $\text{distance}_i(u)$ be distance at step i
- For step i,

  $\text{distance}_i(w) = \text{Min}(\text{distance}_{i-1}(w), \text{distance}_{i-1}(u) + 1)$
- By induction,

  $d(w) \leq \text{distance}_{i-1}(w)$ and $d(u) \leq \text{distance}_{i-1}(u)$
- $d(w) \leq d(u) + 1 \leq \text{distance}_{i-1}(u)+1$
- Therefore, $d(w) \leq \text{distance}_i(w)$

Why Correct?

- Let d(u) be correct distance to u
- Claim 1: All nodes have distance(u)≥d(u)
- Claim 2: After processing a node w, all processed nodes u have distance(u)=d(u)≤d(w), all unprocessed nodes u’ have d(u’)≥d(w), and for all u, distance(u) is the length of the shortest path from v to u where intermediate nodes are constrained to be processed

Proof

- True at beginning. Inductively assume true for the first i-1 nodes we process. Now process w
- If distance(w) > d(w), then the shortest path to w goes contains nodes that haven't been processed
- Let u be the first such node
- All on shortest path to u have been processed
- $\text{distance}(u) = d(u) \leq d(w) < \text{distance}(w)$
- Therefore, u would have been processed instead of w
- Would have set $\text{distance}(w) \leq d(u)+1=d(w)$

Proof

- Say we are processing w and update w’
- Let dist be length of shortest path to w’ through processed nodes, u be last node on this path
- If w is on path, u = w (otherwise, since d(u)≤d(w), we can bypass bypass w without increasing distance)
  - distance(w’) = distance(w)+1
- If w is not on path, that distance(w’) was already set correctly
- Thus distance(w’) = Min of these two cases

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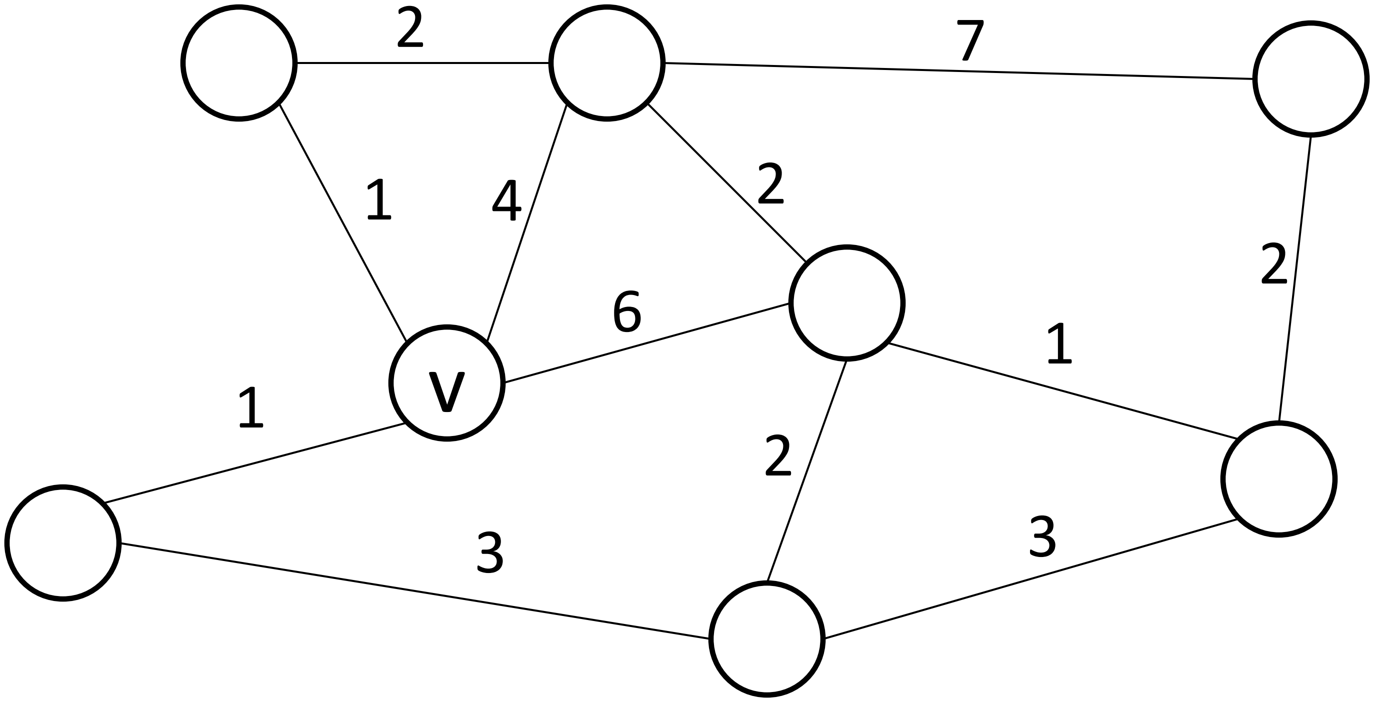
- Therefore, once all nodes are processed, distances correct

Weighted Edges

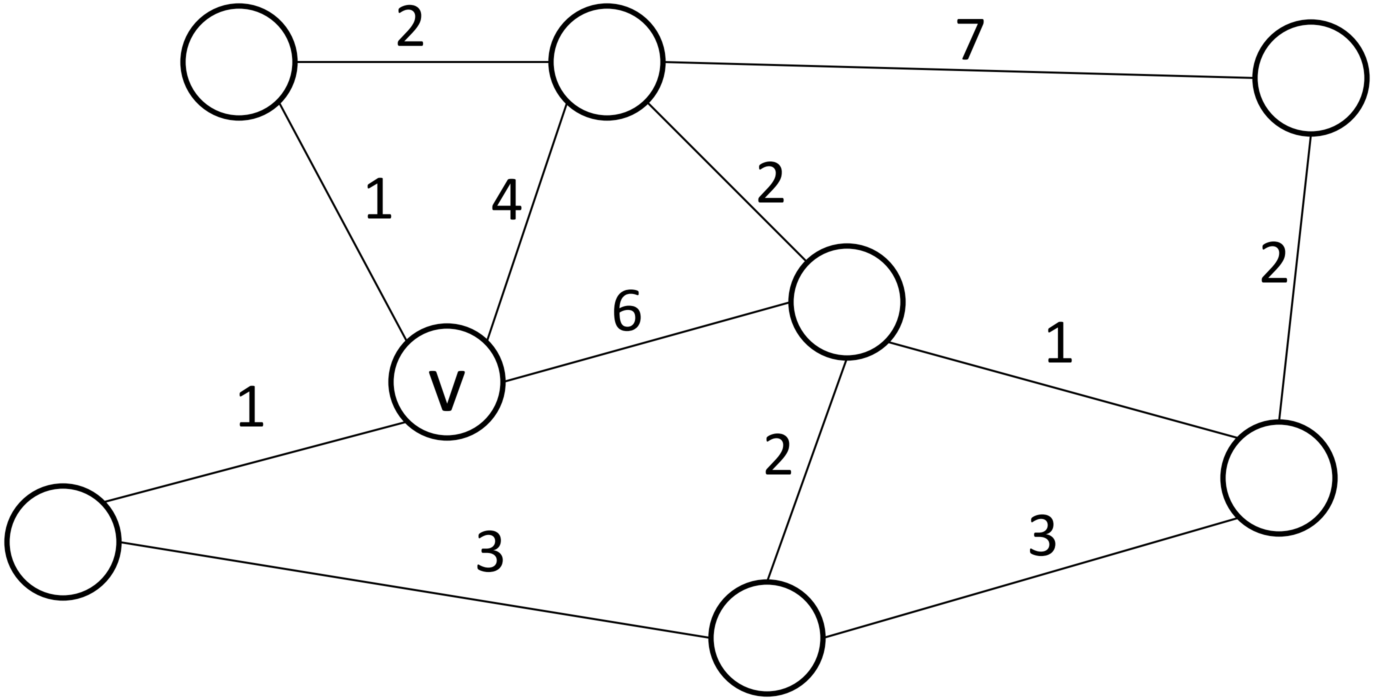


Weighted Edges

- Length of path = sum of weights of edges
- Shortest path = path with shortest length
- Distance from v to u = length of shortest path from v to u

Shortest Path with Positive Integer Weights

- Idea: replace edge of length k with k-1 nodes and k unweighted edges



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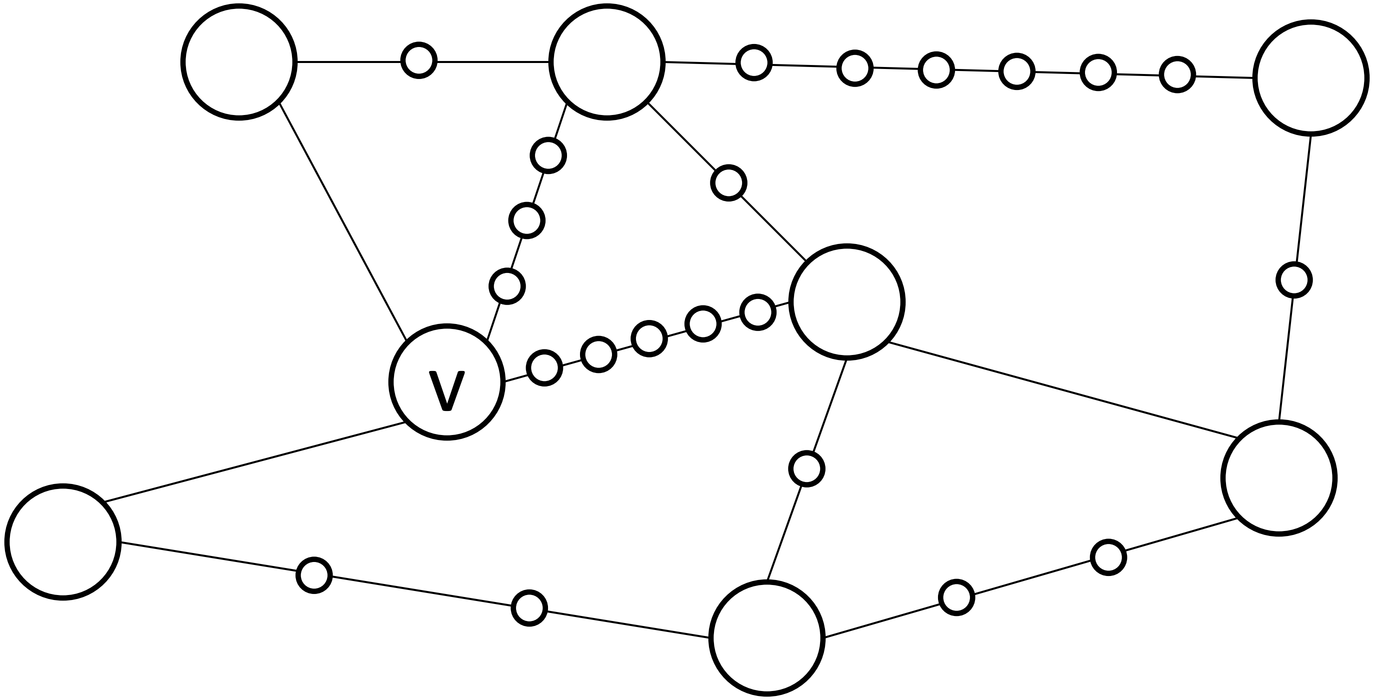

Running Time?

- Let W be total weight
- |V’|:
  - |V| + extra nodes
  - Edge of weight w gets w-1 extra nodes
  - Sum over all edges: W-|E|
  - |V’| = |V|-|E|+W
- |E’| = W
- O(|V’|+|E’|) = O(|V|+W)

Problem

- Can’t handle non-integer weights
- W can be very large – poor performance

Solution: Dijsktra's Algorithm

- Recall BFS high level idea:
  - Set distance(u) = ∞ for u ≠ v
  - Set distance(v) = 0
  - Repeatedly process nodes:
    - Find closest node u that hasn't been processed
    - For each edge (u,w), update distance to w
- Works for general (non-negative) edge weights!
  - update sets distance(w)=Min(distance(w),distance(u)+weight(u,w))
- Issue: now distances might be updated multiple times, so can't use queue

Why Correct?

- Let d(u) be correct distance to u
- Claim 1: All nodes have distance(u)≥d(u)

Proof

- True at beginning. Inductively assume true for first i-1 updates
- Let $\text{distance}_i(u)$ be distance at step i
- For step i,

  $\text{distance}_i(w) = \text{Min}(\text{distance}_{i-1}(w),\text{distance}_{i-1}(u) + 1)$
- By induction,

  $d(w) \leq \text{distance}_{i-1}(w)$ and $d(u) \leq \text{distance}_{i-1}(u)$
- $d(w) \leq d(u) + \text{weight}(u,w) \leq \text{distance}_{i-1}(u)+\text{weight}(u,w)$
- Therefore, $d(w) \leq \text{distance}_i(w)$

Why Correct?

- Let d(u) be correct distance to u
- Claim 1: All nodes have distance(u)≥d(u)
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- Therefore, u would have been processed instead of w
- Would have set distance(w) ≤ d(u)+weight(u,w)=d(w)

Proof

- Say we are processing w and update w’
- Let dist be length of shortest path to w’ through processed nodes, u be last node on this path
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- Therefore, once all nodes are processed, distances correct

How to Allow Multiple Updates

- Queue no longer works
- Instead, use a heap!
- Heap contains all unprocessed nodes
  - Ordered by distance
- To find closest, deletemin()
- To update:
  - distance(w) = Min(distance(w),distance(u)+weight(u,w))
  - decreasekey(w)

Solution: Dijsktra’s Algorithm

- Set distance(v) = 0, distance(u) = ∞ for u ≠ v
- Create heap q with all nodes, ordered by distance
- While q is not empty:
  - Let u = q.deletemin()
  - For each edge (u,w) in E:
    - distance(w) = Min(distance(w),distance(u)+weight(u,w))
    - decreasekey(w)

Running Time?

- |V| deletemin operations
- |V|+|E| insert/decreasekey operations (|V|+2|E| in undirected graphs)
- Recall heap operations:
  - deletemin: O(log |V|)
  - insert/decreasekey: O(log |V|)
- Total time: O((|V|+|E|)log |V|)

Potential Optimization: Lists

- Implement heap operations using a linked list
  - Insert: add beginning of list $O(1)$
  - decreasekey: do nothing $O(1)$
  - deletemin: search whole list for minimum $O(|V|)$
- Total time: $O(|V|^2)$
  - Better for dense graphs $|E|=O(|V|^2)$
  - Worse for sparse graphs

Best of Both Worlds: d-ary Heaps

- Recall d-ary heap operations:
  - deletemin: $O(d \log |V|/\log d)$
  - decreasekey: $O(\log |V|/\log d)$
- Total time:

$$O(|V|\ d \log |V|/\log d + (|V|+|E|) \log |V|/\log d)$$

$$= O(\ (d\ |V|+|E|)\log |V|/\log d)$$

- What d minimizes this quantity?

Choosing d

- Minimize $O((d\ |V|+|E|)\log |V|/\log d)$
- Difficult to minimize exactly
- Possible to show that $d = O(|E|/|V|)$ is optimal
- Some cases:
  - Sparse: $|E|=O(|V|)$, $d=O(1)$, time=$O(|V| \log |V|)$
    - Same as with binary heaps
  - Dense: $|E|=O(|V|^2)$, $d=O(|V|)$, time=$O(|V|^2)$
    - Same as with linked lists
  - Intermediate: $|E|=O(|V|^{1+c})$, $d=O(|V|^c)$, time= $O(|V|^{1+c})$
    - Linear, better than both binary heaps and linked lists

Even Better: Fibonacci Heap

- Complicated data structure
  - deletemin: O(log |V|)
  - insert/decreasekey: O(1) amortized
- Total running time: O(|E|+|V|log |V|)
  - Only asymptotically better when |E| is close to |V|, but not O(|V|)
  - Example: |E|=O(|V|log |V|)

Finding Actual Path

- So far, we only compute distance from v to other nodes
- How do we compute the actual shortest path?
- If processing node u causes last change to distance(w), some shortest path to w passes through u
- Keep track of prev(w), the previous node in shortest path

Finding Actual Path

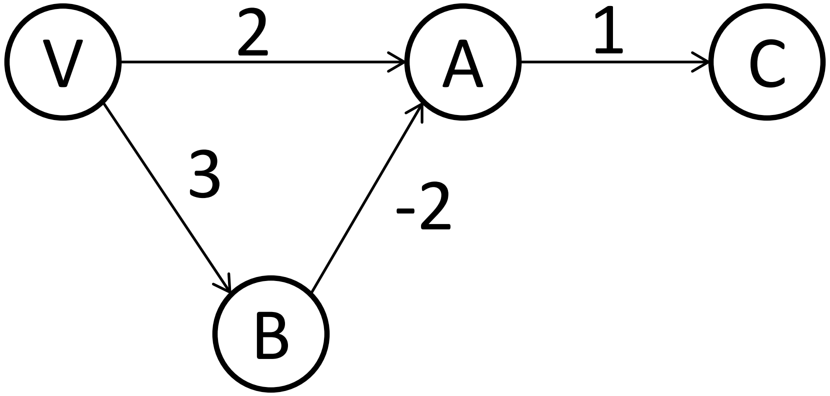
- Set distance(v) = 0, distance(u) = ∞ for $u \neq v$
- Create heap q with all nodes, ordered by distance
- While q is not empty:
  - Let u = q.deletemin()
  - For each edge (u,w) in E:
    - If(distance(u)+weight(u,w) < distance(w)):
      - distance(w) = distance(w)+weight(u,w)
      - prev(w) = u

Finding Actual Path

- To find path from v to w, follow prev pointers form w back to v

Negative Edges

- If we have negative edges, Dijsktra’s algorithm fails



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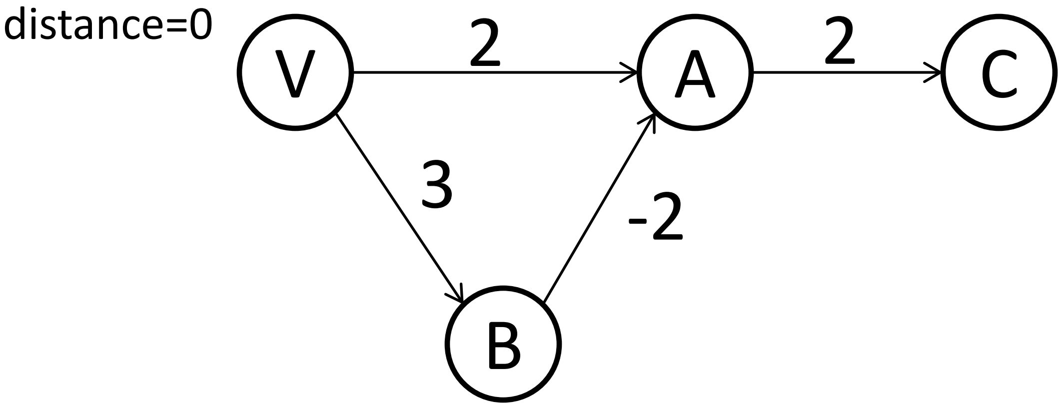

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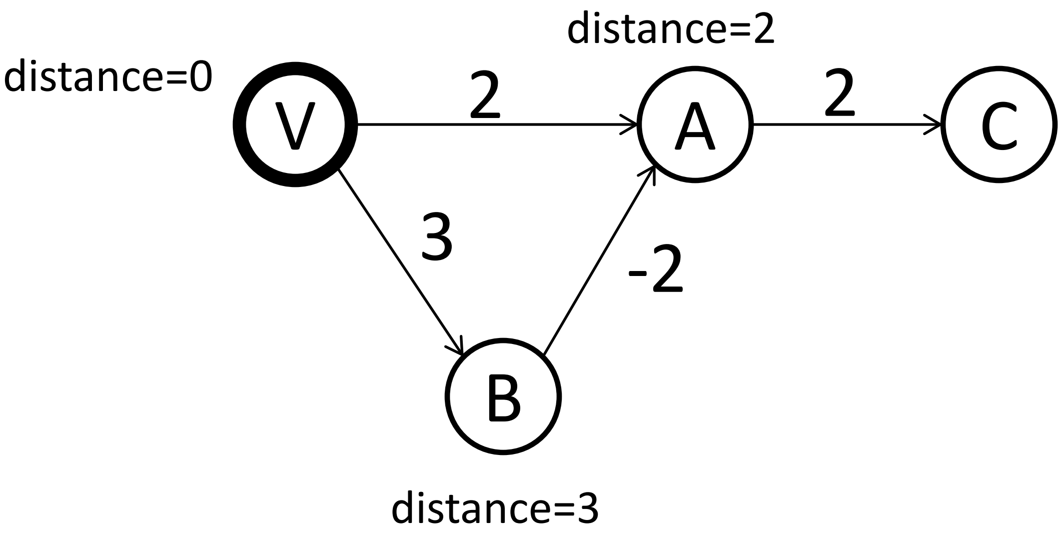

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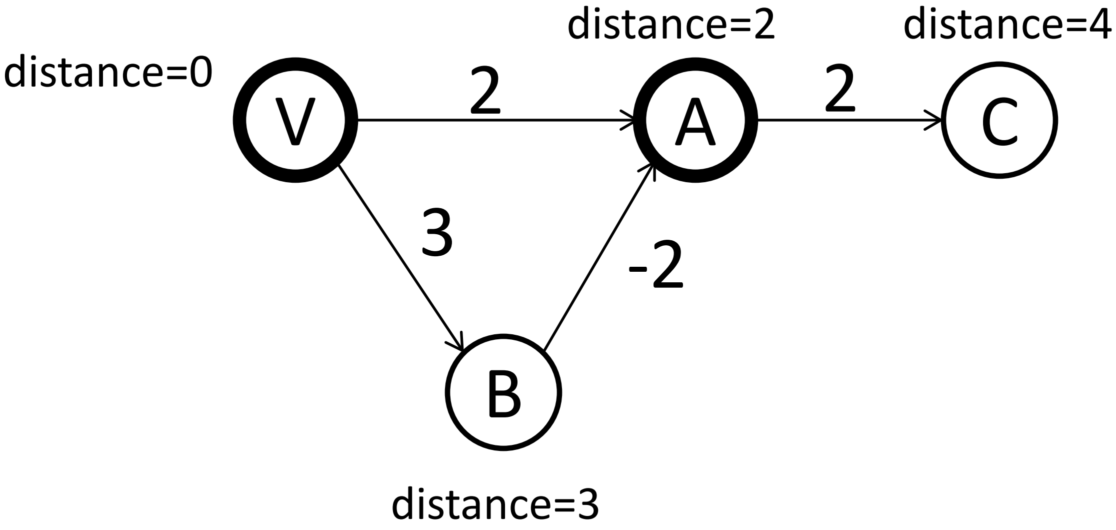

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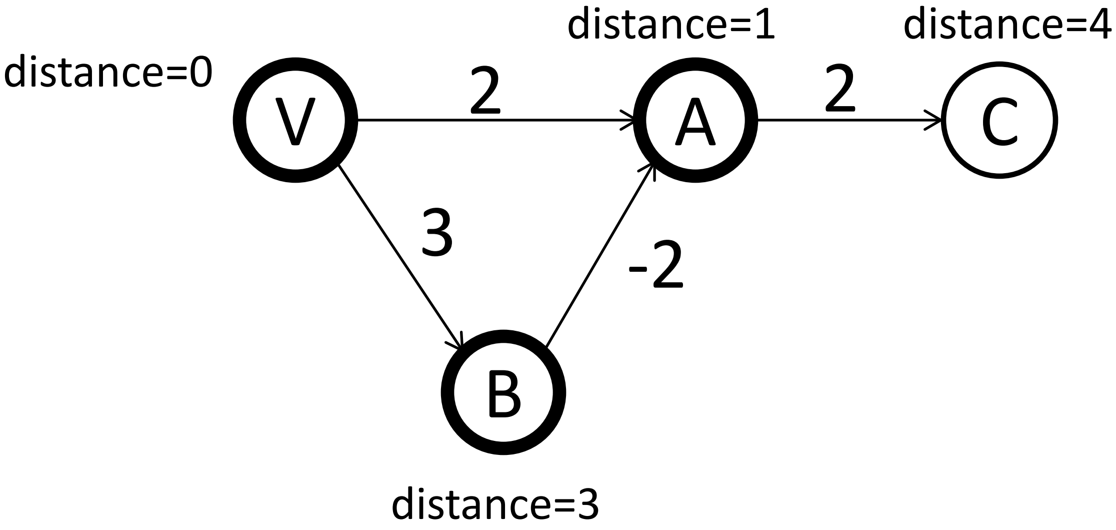

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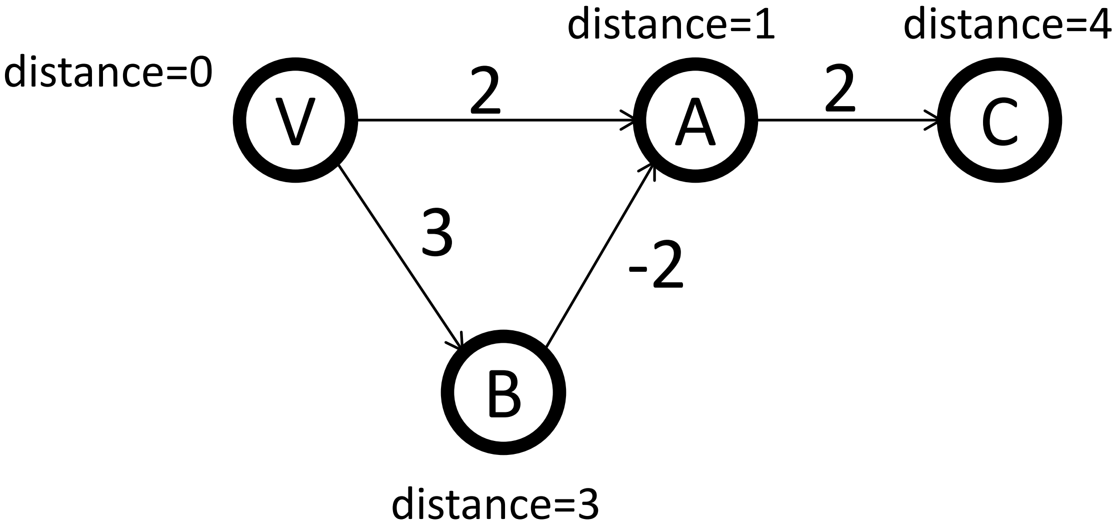

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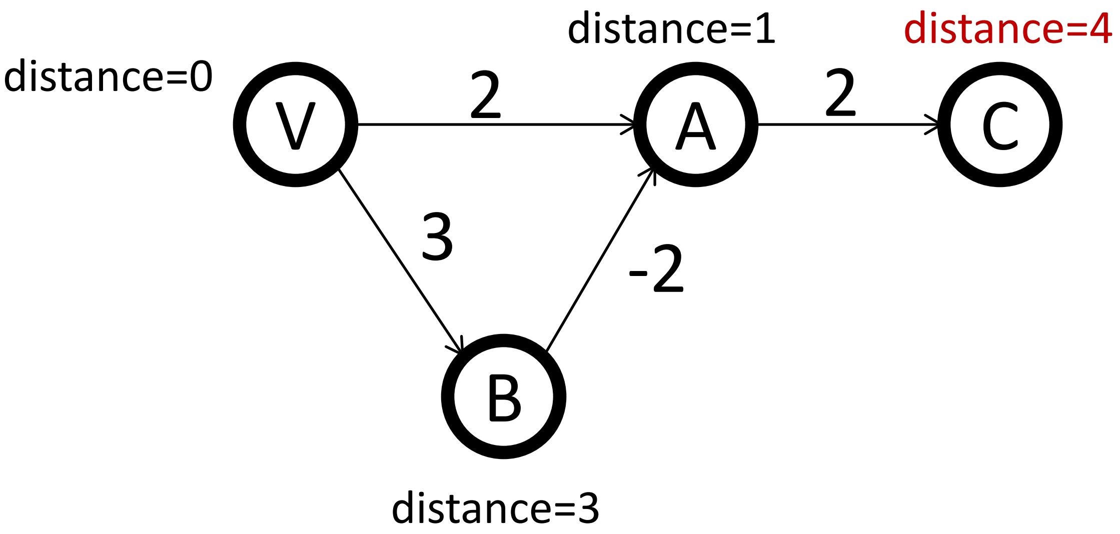

- Distance from V to C is 3, not 4!

Why Fail on Negative Edges?

- Let d(u) be correct distance to u
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Why Fail on Negative Edges

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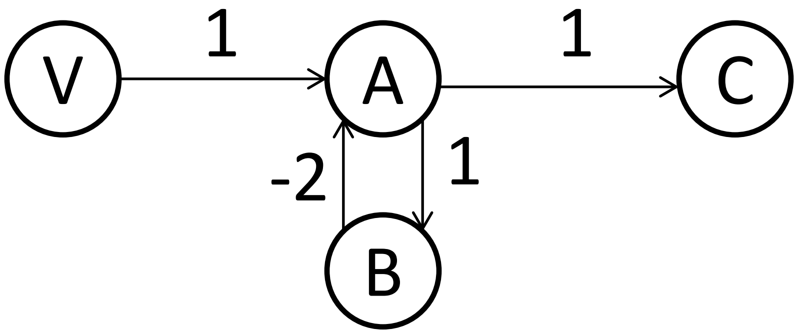
- distance(u) = d(u) ≤ d(w) < distance(w)
- Therefore, u would have been processed instead of w
- Would have set distance(w)≤d(u)+weight(u,w)=d(w)

Bigger Problem: Negative Cycles

- In the presence of a negative cycle, the shortest path problem is undefined



- Path VAC has length 2
- Path VABAC has length 1
- Path VABABAC has length 0
- Path VABABABAC has length -1 …

Negative Cycles

- If negative cycles, no shortest path
- Possible solution: add constant value to every edge
  - Does not compute shortest paths!
- Possible solution: shortest simple path
  - Finite number of simple paths, so shortest exists
  - Turns out to be very hard in presence of negative cycles