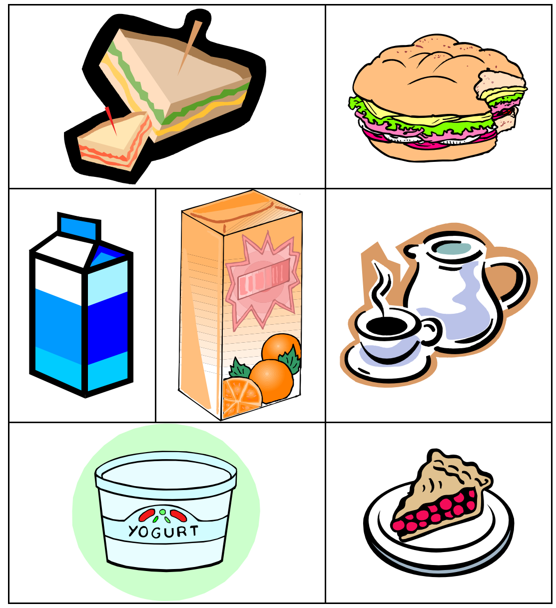
The Fundamental Counting Principle

Many problems in probability and statistics require careful analysis of complex events. Combinatorics' basic roots are to develop systematic ways of counting. These systematic counting methods will allow the solving of complex counting problems that are useful in all facets of life.



Investigate: Counting Without Counting

1. A café has a lunch special consisting of an egg or a ham sandwich (E or H); milk, juice, or coffee (M, J, or C); and yogurt or pie for dessert (Y or P).

   a) One item is chosen from each category. List all possible meals. Use a **tree diagram** to organize your work.

   b) How many possible meals are there? Count the ends of the branches of your tree diagram.

   c) How can you determine the number of possible meals without listing all of them?

2. The cafe also features ice cream cones in 24 flavours, and 6 toppings. You can order regular, sugar or waffle cones. How many possible ice cream cones can you order?

The Fundamental Counting Principle:

- If we can perform a first task in *x* different ways
- If we can perform a second task in *y* different ways
- If we can perform a third task in *z* different ways, and so on . . .

Then the first task followed by the second and so on can be performed in $x \cdot y \cdot z$ . . . different ways.

Example 1: A computer store sells 6 different computers, 4 different monitors, 5 different printers, and 3 different multimedia packages. How many different computer systems are available?

Example 2: How many different 2-digit numbers are there?

Counting objects with restrictions

We will continue to use the fundamental counting principle and use "blanks" instead of trees, but it is important that we count the restricted value first!

Example 3: In each case, how many 2-digit numbers can be formed using the digits 0, 1, 2, 3, and 4?

a) Repetition of digits is allowed.

b) Repetition of digits is not allowed.

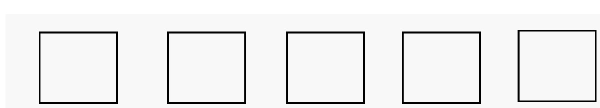
Example 4: A license plate consists of 3 letters followed by 3 digits. Determine the total number of possible license plates if the following conditions apply:

a) There are no restrictions on letter or digits.

b) No letter or number can be repeated.

Example 5: How many odd 3 digit numbers can be made from the numbers {0,1,2,3,4,5,6}?

Example 6: How many arrangements can be formed using all of the letters of the word MUSIC?



Example 7: Your teacher announces there will be a seating plan change next week. How many possible seating plans can be made if there are 20 students and 20 desks?

n objects in n places: The number of ways of arranging ***n*** objects in ***n*** places is

n x (n–1) x (n–2) x (n–3) . . . 3 x 2 x 1 which we abbreviate as ***n!*** (read "n factorial)

Ex.

0! = 1

1! = 1

2! = 2 *x* 1 = 2

3! = 3 *x* 2 *x* 1 = 6

You'll see later on that sometimes the recursive definition is also useful: $n! = n(n-1)!$

Factorials on the TI-83: **MATH, →, PRB, !**

Ex.

Find 8! on your graphing calculator:

Example 9: 12 people are arranged in a line for a picture, how many different photos can be taken?

Example 10: How many ways can parents assign 5 chores to 5 children?

Specific Positions:

How many ways can Adam, Beth, Charlie, and Doug be seated in a row if Charlie must be in the second chair?

How many ways can you order the letters of KITCHEN if it must start with a consonant and end with a vowel?

How many numbers can be made from rearranging 2345678 if the number must begin with exactly two odd digits?

Objects together or apart:

How many arrangements of the word ACTIVE are there if C & E must always be together?

Strategy for objects together,

1) **Treat the objects together as 1, determine the number of arrangements**
2) **For each group that is together, find the number of "internal" arrangements**

Strategy for objects apart,

How many ways can 3 math books, 5 chemistry books, and 7 physics books be arranged on a shelf if the books of each subject must be kept together?

Johnny and Billy fight when they sit together. If there are 11 people in their day care, how many ways can they line up for a photo is Johnny and Billy must be apart?

Practice

1) There are 6 people in a race. In how many ways can they finish first, second or third ?

2) A golfer has 4 different hats, 3 gloves and 2 pairs of shoes to pick from for his round of golf. In how many ways can he make his choices ?

3) In Canada, postal codes consist of 6 characters -- three letters and three digits. Each postal code starts with a letter and alternates with a digit.
   a. How many postal codes are there ?
   b. How many start with the letter S ?
   c. How many start with the letter S and end in the digit 8 ?
   d. How many start with the letter S, digit 6 and NO letter or digit is repeated ?

4) Using the digits { 1 , 2 , 3 , 4 , 5 }, how many positive three digit integers can be made if:
   a. there are NO restrictions
   b. it is odd and repetition is allowed ?
   c. it is odd and repetition is NOT allowed ?
   d. Repeat question a, b and c if the digits you can choose are { 0 , 1 , 2 , 3 , 4 , 5 }.

5) In how many ways can ALL of the letters of the word TRAVEL be arranged if:
   a. there are NO restrictions ?
   b. it must start with T ?
   c. it starts with a consonant and ends in a vowel ?
   d. the letters TR must stay together ?

6) How many positive even three-digit integers less than 400 can be formed from the digits { 0 , 1 , 2 , 3 , 4 , 5 } if:
   a. repetition is allowed ?
   b. No digit is repeated ?

7) You are ordering dinner at a restaurant. How many ways can you order a meal if you have two choices for a drink (coffee or tea), three main courses to choose from (chicken, beef, or fish) and two desserts (pie or cake) ?
   a. Draw a tree diagram
   b. Use the fundamental counting principle

8) Eight sprinters are in the final of a race. How many different ways there to award the gold, silver and bronze medals ?

9) Television stations in Canada usually have call letters that are 4 letters long and begin with the letter C. If the CRTC made this a law in Canada, then how many television stations could the CRTC license ?

10) Repeat the above question using the restriction, repetition of letters is NOT allowed

11) Some license plates consist of 3 letters followed by 3 numbers. How many different license plates are possible if:
   a. if there are NO Restrictions
   b. if the letters must be DIFFERENT
   c. if the letters are different and the first digit can't be 0

12) How many two digit whole numbers can be formed using the digits: 0,1,2,4,6,7,8,9 (8 digits) ?
   a. Repetitions are allowed
   b. Repetitions are not allowed

13) An ice cream parlor features 64 flavors and 20 toppings, in 3 sizes. How many different sundaes can be made ?

14) How many EVEN two digit numbers are there ?

15) How many EVEN two digit numbers can be made using the digits 1 , 2 , 3, 4, 5, 6, 7, 8 ?
   a. Repetitions are not allowed
   b. Repetitions are allowed

16) How many two digit numbers can be formed using the digits 0 , 2 , 4 , 6 , 8 if:
   a. Repetitions are allowed
   b. Repetitions are not allowed

17) How many ODD four digit numbers can be made from all of the digits, if:
   a. Repetition is allowed
   b. Repetition is not allowed

18) In how many ways can all of the letters of the word PROBLEM be arranged ?

19) In how many ways can all of the letters of the word PROBLEM be arranged if the arrangement must start with a consonant and end in a vowel ?

20) How many ways can the letters in the word PENCIL be arranged?

21) If there are four different types of cookies, how many ways can you eat all of them?

22) If three albums are placed in a multi-disc stereo, how many ways can the albums be played?

23) How many ways can you order the letters in KEYBOARD if K and Y must always be kept together?

24) How many ways can the letters in OBTUSE be ordered if all the vowels must be kept together?

25) How many ways can 4 rock, 5 pop, & 6 classical albums be ordered if all albums of the same genre must be kept together?

HW: MC 4,10,20

Key

1)

3 decisions

3 blanks → $\underline{6} \cdot \underline{5} \cdot \underline{4} = 120$

- 6: Any of the 6 people.
- 5: The runner who finishes first can't finish second.
- 4: Must be 4 runners left since first and second are picked.

2)

3 decisions

3 blanks → $\underline{4} \cdot \underline{3} \cdot \underline{2} = 48$

- 4: 4 hats
- 3: 3 gloves
- 2: 2 pairs of shoes

3a)

Letter, Letter, Letter [26 letters in the alphabet]

6 decisions

6 blanks → $\underline{26} \cdot \underline{10} \cdot \underline{26} \cdot \underline{10} \cdot \underline{26} \cdot \underline{10} = 17{,}576{,}000$

digit, digit, digit

There are 10 digits { 0 , 1 , 2 , 3 ▪ ▪ ▪ 9}

b)

Must be S; Any letter [26 letters in the alphabet]

6 decisions

6 blanks → $\underline{1} \cdot \underline{10} \cdot \underline{26} \cdot \underline{10} \cdot \underline{26} \cdot \underline{10} = 676{,}000$

digit, digit, digit

There are 10 digits { 0 , 1 , 2 , 3 ▪ ▪ ▪ 9}

c)

Must be S; Any letter [26 letters in the alphabet]

6 decisions

6 blanks → $\underline{1} \cdot \underline{10} \cdot \underline{26} \cdot \underline{10} \cdot \underline{26} \cdot \underline{1} = 67{,}000$

Look at Restrictions first

digit, digit

Must be the digit 8. Only one way to get the digit 8 d)

Must be S   S can't be used so 25 letters remain, then 24

6 decisions

6 blanks → $\underline{1} \cdot \underline{1} \cdot \underline{25} \cdot \underline{9} \cdot \underline{24} \cdot \underline{8} = 43,200$

Look at Restrictions first

Must be 6 One choice

digit 6 is used, therefore 9 digits remain to pick from

Now 8 digits remain

4)

a)

3 blanks → $\underline{5} \cdot \underline{5} \cdot \underline{5} = 125$

Each blank can be any of the 5 digits

b)

3 blanks → $\underline{5} \cdot \underline{5} \cdot \underline{3} = 75$

Any of the five digits

Only the digits 1, 3, 5 are odd, so 3 choices

c)

Fill in the restrictions first !

Only 3 digits are left for the middle spot

3 blanks → $\underline{4} \cdot \underline{3} \cdot \underline{3} = 36$

Only 4 left since either 1, 3, or 5 is used at the end and you can't repeat digits

Only the digits 1, 3, 5 are odd, so 3 choices

d part a)

3 blanks → $\underline{5} \cdot \underline{6} \cdot \underline{6} = 180$

Cannot be 0 Since it would NOT be a 3 digit integer

Any of the 6 digits

d part b)

Can be any of the 6 digits ↓

3 blanks → $\underline{5} \cdot \underline{6} \cdot \underline{3} = 90$

↑ Cannot be 0
So, 5 digits
to pick from

↑ Only the digits 1, 3, 5
are odd, so 3 choices

d)

(c) it is odd and repetition is NOT allowed ?

Fill in the restrictions first !

2 digits are used from the 6
Therefore, 4 digits to pick from ↙

3 blanks → $\underline{4} \cdot \underline{4} \cdot \underline{3} = 48$

↑ Cannot be 0
and either
1, 3, or 5 is
used at the end
Therefore, 4 digits
remain

↑ Only the digits 1, 3, 5
are odd, so 3 choices

5a)

6 blanks → $\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 6! = 720$

Note : All letters must be used → NO Repetition

b)

6 blanks → $\underline{1} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5! = 120$

↑ One way for T
to be picked

c)

Remember → Fill in the restrictions first

4 letters are left for these 4 blanks ↙ ↓ ↓ ↘

6 blanks → $\underline{4} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{2} = 192$

↑ 4 consonants
to pick from

↑ 2 vowels to
pick from

d)

Think of TR as one big blank

↓

2! ways you can arranging TR

↓

$\underline{\quad\quad} \cdot \underline{\ } \cdot \underline{\ } \cdot \underline{\ } \cdot \underline{\ }$

↓

Now, arrange these 4 blanks with the big TR blank → 5 blanks

↓ ↓

$2! \cdot 5! = 240$

6a)

Any of the 6 can be used

↓

$\underline{3} \cdot \underline{6} \cdot \underline{3} = 54$

↑ ↑

Must be less than 4 {0,1,2,3} but can't be 0. So, 3 choices

Must be even {0,2,4} can be used. 3 choices

b)

Remember, fill in restrictions first, but in this question you have overlapping restrictions.

(restrictions that affect the other restrictions)

Digits less than 4: 0 , 1 , 2 , 3 , 4 , 5

Digits that are even: 0, 2, 4

So, break up the question into parts – this will eliminate a restriction.

Last digit is 4 or Last digit is 2 or Last digit is 0

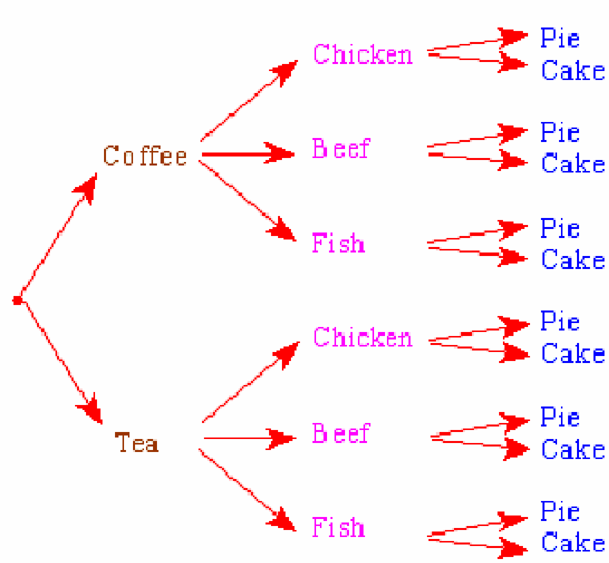
$\underline{3} \cdot \underline{4} \cdot \underline{1} + \underline{2} \cdot \underline{4} \cdot \underline{1} + \underline{3} \cdot \underline{4} \cdot \underline{1} = 12+8+12$

$= 32$

- Last digit is 4: Must be 1 , 2 or 3. Can't be 0 — 4 digits left
- Last digit is 2: Must be 1 or 3 — 4 digits left
- Last digit is 0: Must be 1, 2 or 3 — 4 digits left

7a)

Draw a picture — this is known as a TREE DIAGRAM



Follow each path; we end up with 12 different meals that you can order.

b)

Drawing a picture each time would be a lot of work

Let's use the Fundamental Counting Principle

Think of this question as making 3 choices.
So, lets use 3 blanks.

Multiply

$\underline{2} \cdot \underline{3} \cdot \underline{2} = 12$

- There are 2 choices for a drink
- 3 choices for main
- 2 choices for dessert

8)

3 medals → 3 blanks

Gold, Silver, Bronze

$\underline{8} \cdot \underline{7} \cdot \underline{6} = 336$

- 8 could finish first
- 7 are left to finish second
- 6 are left to finish third

9)

4 letters must be used therefore 4 choices to make

4 blanks

$\underline{1} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 17{,}576$

- Must begin with C, so only 1 to pick from
- There are 26 letters in the alphabet, so 26 choices for each blank. NOTE: The question did not say you couldn't repeat letters

10)

$\underline{1} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 13{,}800$

- Must be the letter C
- Can't be a C
- 2 are used only 24 left
- 3 are used so only 23 are left

11)

6 choices → 6 blanks

3 letters, 3 numbers

a) $\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 17{,}576{,}000$

b) $\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 15{,}600{,}000$

c) $\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{9} \cdot \underline{10} \cdot \underline{10} = 14{,}040{,}000$

12)

(a) repetitions are allowed

8 digits

$\underline{7} \cdot \underline{8} = 56$

Can be any of the 8 digits

Can't be 0, because it would not be a 2 digit number

(b) repetitions are NOT allowed

$\underline{7} \cdot \underline{7} = 49$

Can't be 0

Can be 0 but NOT the one picked in first blank

NOTE

Which blank do you start with?

— the one that has restrictions.

— like can't be 0

13)

$\underline{64} \cdot \underline{20} \cdot \underline{3} = 3840$

14)

0,1,2,3,4,5,6,7,8,9 All possible digits only 5 are even

$\underline{9} \cdot \underline{5} = 45$

Can't be 0

Must be even

15)

(a) Repetitions are NOT allowed

Start HERE

$\underline{7} \cdot \underline{4} = 28$

We have used 1 of the 8, 7 left

4 of the digits are EVEN

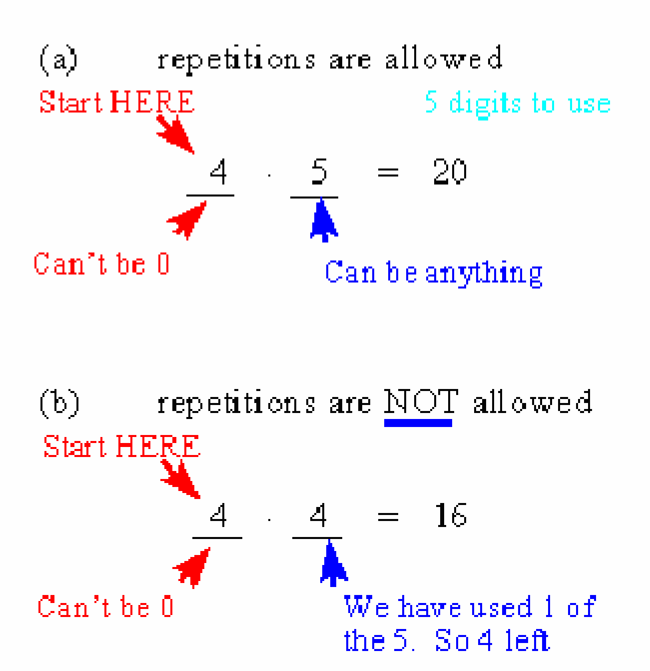
(b) Repetitions are allowed

Start HERE

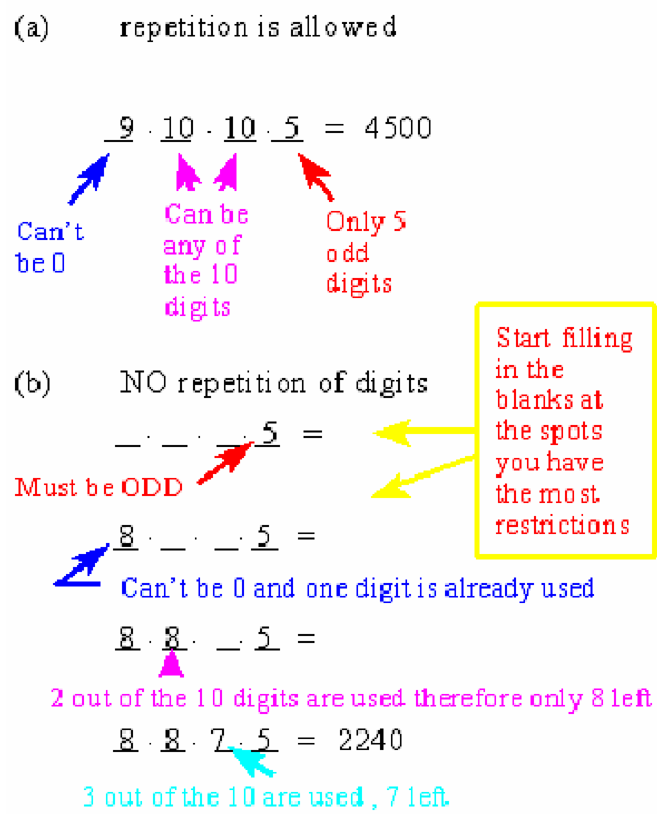
$\underline{8} \cdot \underline{4} = 32$

Any of the 8 digits

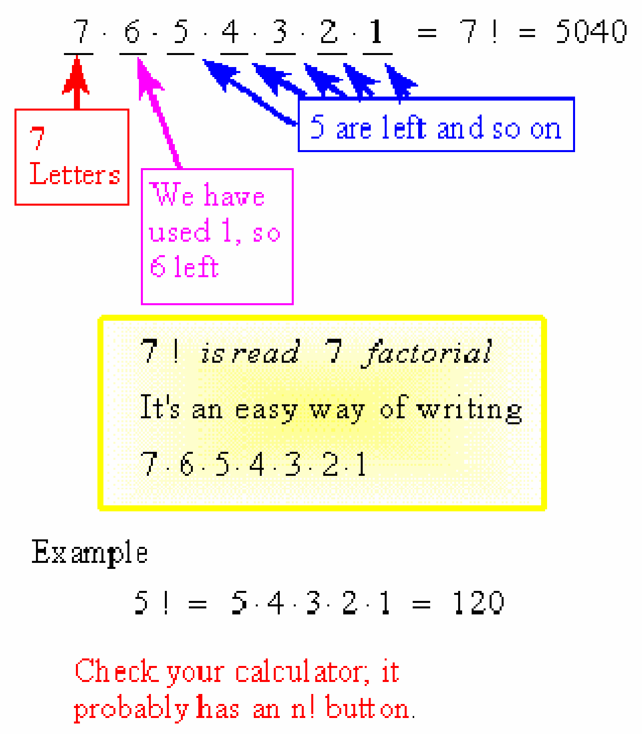
16)



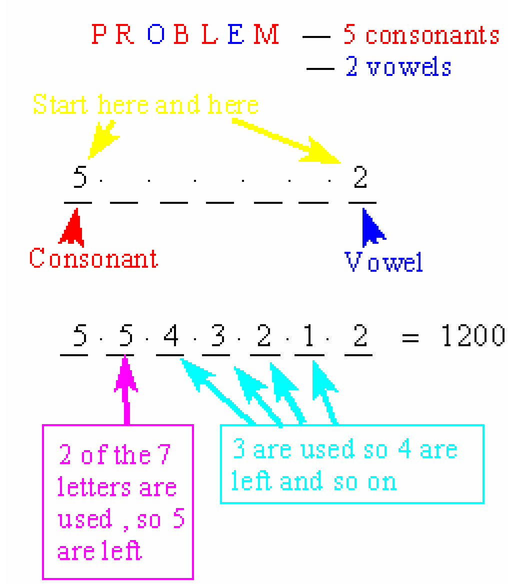
17)



18)



19)



20) 6!
21) 4!
22) 3!
23) 2! 7! = 10080
24) 4! 3! = 144
25) 4! 5! 6! 3! = 12441600