COUNTING PROBLEMS

HOW TO UNDERSTAND AND HOW TO DO ABSOLUTELY EVERYTHING THROWN YOUR WAY IN HIGH SCHOOL ON MATTERS OF COUNTING

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This material comes from chapter 16 of: ***THINKING MATHEMATICS! Vol2: Advanced Counting and Advanced Algebra Systems***

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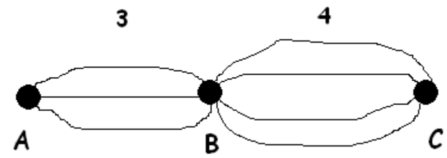
STEP 1: THE MULTIPLICATION PRINCIPLE

Let's start with a tiny puzzle:

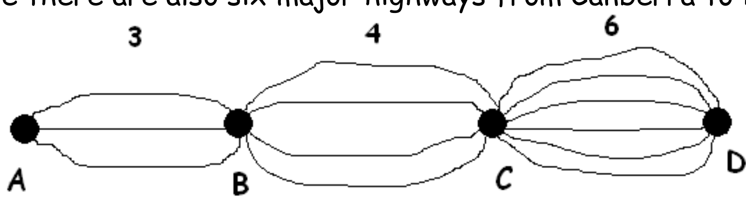
There are three major highways from Adelaide to Brisbane, and four major highways from Brisbane to Canberra.



How many different routes can one take to travel from Adelaide to Canberra?

Thinking Exercise 1: Is the answer **7** (from $3+4$) or is the answer **12** (from 3×4)? Be very clear in your own mind as to whether one should add or should multiply these numbers.

Suppose there are also six major highways from Canberra to Darwin.



How many different routes are there from A to D?

Thinking Exercise 2: Explain, very carefully, why the answer is $3\times 4\times 6 = 72$.

I own five different shirts, four different pairs of trousers and two sets of shoes. How many different outfits could you see me in?

Exercise 3: And the answer is …?

There are ten possible movies I can see and ten possible snacks I can eat whilst at the movies. I am going to see a film tonight and I will eat a snack. How many choices do I have in all for a movie/snack combo?

Exercise 4: And the answer is …?

We have a general principle:

THE MULTIPLICATION PRINCIPLE
If there are a ways to complete a first task and b ways to complete a second task, and no outcome from the first in any way affects a choice of outcome from the second, then there are $a \times b$ ways to complete both tasks as a pair.

This principle readily extends to the completion of more than one task.

Thinking Exercise 5: Explain the clause stated in the middle of the multiplication principle. What could happen if different outcomes from the first task affect choices available for the second task? Give a concrete example.

EXAMPLE: On a multiple choice quiz there are five questions, each with three choices for an answer:

1. A B C
2. A B C
3. A B C
4. A B C
5. A B C

I decide to fill out my answers randomly. In how many different ways could I fill out the quiz?

Answer: This is a five-stage process:

Task 1: Answer question one: 3 ways
Task 2: Answer question two: 3 ways
Task 3: Answer question three: 3 ways
Task 4: Answer question four: 3 ways
Task 5: Answer question five: 3 ways

By the multiplication principle there are $3 \times 3 \times 3 \times 3 \times 3 = 3^5$ ways to complete the quiz. □

That's all there is to say about this first step to counting!

STEP 2: WORD GAMES

My name is JIM. In how many ways can I arrange the letters of my name?

Answer 1: We could just list the ways.

JIM MJI IJM
JMI MIJ IMJ

There are six ways.

Answer 2: Use the multiplication principle: We have three slots to fill:

___ ___ ___

The first task is to fill the first slot with a letter. There are 3 ways to complete this task.

The second task is to fill the second slot. There are 2 ways to complete this task. (Once the first slot is filled, there are only two choices of letter to use for the second slot.)

The third task is to fill the third slot. There is only 1 way to complete this task (once slots one and two are filled).

$$\underline{3}\ \underline{2}\ \underline{1}$$

By the multiplication principle, there are $3\times 2\times 1=6$ ways to complete this task.

□

In how many ways can one arrange the letters HOUSE ?

Exercise 6: And the answer is …?

In how many ways can one arrange the letters BOVINE ?

Exercise 7: And the answer is …?

When playing with these problems it is clear that the following definition is needed:

Definition: The product of integers from 1 to N is called N *factorial* and is denoted $N!$.

These factorial numbers grow very large very quickly:

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$
$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$
$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

What is the first factorial larger than a million? A billion?

ON A CALCULATOR there is a Factorial feature hidden under the MATH button under the PROBABILITY menu.

What is the largest factorial your calculator can handle?

One more question ..

In how many ways can one arrange the letters FACETIOUSLY ?

Exercise 8: And the answer is …?

I ask this extra question simply as an excuse to pose an interesting non-math question:

What is unusual about the vowels (including y) of facetiously?

As far as I am aware there is only one other word in the English language with this property! What is that word?

So far all is fine and dandy. We are lucky that my name isn't BOB. How might we handle repeated letters?

How many ways are there to arrange the letters BOB? Assume the Bs are indistinguishable?

Exercise 9: Answer this question! Do you think the only way to answer it is to list all the possibilities? (It is probably the easiest way.)

In how many ways can one arrange the letters HOUSES?

Thinking Exercise 10: Try answering this one. Does listing all the possibilities seem fun?

Here's a clever way to think of this problem …

If the Ss were distinguishable – written, say, as S_1 and S_2 - then the problem is easy to answer:

There are $6! = 720$ ways to rearrange the letters $HOUS_1ES_2$.

The list of arrangements might begin:

$HOUS_1ES_2$
$HOUS_2ES_1$

$OHUS_1S_2E$
$OHUS_2S_1E$

S_1S_2UEOH
S_2S_1UEOH
⋮

But notice, if the Ss are no longer distinguishable, then pairs in this list of answers "collapse" to give the same arrangement. We must alter our answer by a factor of two and so the number of arrangements of the word HOUSES is:

$$\frac{6!}{2} = 360$$

How many ways are there to rearrange the letters of the word CHEESE?

Answer: If the three Es are distinct – written E_1, E_2, and E_3, say – then there are $6!$ ways to rearrange the letters $CHE_1E_2S\,E_3$. But the three Es can be rearranged $3! = 6$ different ways within any one particular arrangement of letters. These six arrangements would be seen as the same if the Es were no longer distinct:

$$\begin{matrix} HE_1E_2SCE_3 & HE_3E_1SCE_2 & & \\ HE_1E_3SCE_2 & HE_3E_2SCE_1 & \rightarrow & HEESCE \\ HE_2E_1SCE_3 & HE_2E_3SCE_1 & & \end{matrix}$$

Thus we must divide our answer of $6!$ by $3!$ to account for the groupings of six that become identical. There are thus $\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 6 \cdot 5 \cdot 4 = 120$ ways to arrange the letters of CHEESE. □

COMMENT: The number of ways to rearrange the letters HOUSES is $\frac{6!}{2!}$. The "2" on the denominator is really 2!.

Explain why the number of ways to arrange the letters of the word CHEESES is $\frac{7!}{3!2!}$.

Answer: If the Es were distinguishable and the Ss were distinguishable, then we'd be counting the ways to arrange seven distinct letters:

There are $7!$ ways to arrange the letters of $CHE_1E_2S_1E_3S_2$

As before there are $3!$ways to arrange the Es in any particular configuration. These groups $3!$ will "collapse" to the same arrangement if we remove the subscripts from the Es.

$$\begin{matrix} HE_1E_2S_1CE_3S_2 & HE_3E_1S_1CE_2S_2 & & \\ HE_1E_3S_1CE_2S_2 & HE_3E_2S_1CE_1S_2 & \rightarrow & HEES_1CE\,S_2 \\ HE_2E_1S_1CE_3S_2 & HE_2E_3S_1CE_1S_2 & & \end{matrix}$$

$$\begin{array}{ll} HE_1E_2S_2CE_3S_1 & HE_3E_1S_2CE_2S_1 \\ HE_1E_3S_2CE_2S_1 & HE_3E_2S_2CE_1S_1 \quad \rightarrow \quad HEES_2CE\,S_1 \\ HE_2E_1S_2CE_3S_1 & HE_2E_3S_2CE_1S_1 \end{array}$$

Etc.

But these new arrangements also collapse in pairs once we remove the subscripts from the Ss.

$$\begin{array}{ll} HE_1E_2S_1CE_3S_2 & HE_3E_1S_1CE_2S_2 \\ HE_1E_3S_1CE_2S_2 & HE_3E_2S_1CE_1S_2 \quad \rightarrow \quad HEES_1CE\,S_2 \\ HE_2E_1S_1CE_3S_2 & HE_2E_3S_1CE_1S_2 \end{array}$$

$$\rightarrow HEESCES$$

$$\begin{array}{ll} HE_1E_2S_2CE_3S_1 & HE_3E_1S_2CE_2S_1 \\ HE_1E_3S_2CE_2S_1 & HE_3E_2S_2CE_1S_1 \quad \rightarrow \quad HEES_2CE\,S_1 \\ HE_2E_1S_2CE_3S_1 & HE_2E_3S_2CE_1S_1 \end{array}$$

So we need to take our answer $7!$ and divide by $3!$ and divide by $2!$:

$$\frac{7!}{3!} \div 2!$$

BE SURE TO UNDERSTAND WHY THIS EQUALS $\frac{7!}{3!2!}$

(which is also much easier to read!)

In how many ways can one arrange the letters CHEEEEEESIESSTT?

Exercise 11: Be sure to understand why the answer is $\frac{14!}{6!3!2!}$.

Consider the "word" CHEESIESTESSNESS, the quality of being the cheesiest of cheeses. Do you see that there are

$$\frac{16!}{5!6!}$$

ways to arrange its letters?

Exercise 12: Actually evaluate this number.

This may seem strange, but it is actually better to write this answer as:

$$\frac{16!}{1!1!5!6!1!1!1!}$$

$1!$ for the one letter C
$1!$ for the one letter H
$5!$ for the five letters E
$6!$ for the four letters S
$1!$ for the one letter I
$1!$ for the one letter T
$1!$ for the one letter N

This offers a self check: The numbers appearing on the bottom should sum to the number appearing on the top.

Question 13: Does $1!$ make sense?

How many ways can you rearrange the letters of your full name?

FOR THE BOLD: There are no Ps in the word CHEESIESTESSNESS. Does this mean we should write the answer as $\frac{16!}{1!1!5!6!1!1!1!0!}$? There are also no Qs and no Ms: $\frac{16!}{1!1!5!6!1!1!1!0!0!0!}$.

The quantity $0!$ doesn't actually make sense, but what value might we as a society assign to it so that the above formulas still make sense and are correct?

EXTRA:

Exercise 14: Evaluate the following expressions:

a) $\frac{800!}{799!}$ b) $\frac{15!}{13!2!}$ c) $\frac{87!}{89!}$

Exercise 15: Simplify the following expressions as far as possible:

d) $\frac{N!}{N!}$ e) $\frac{N!}{(N-1)!}$ f) $\frac{n!}{(n-2)!}$ g) $\frac{1}{k+1}\cdot\frac{(k+2)!}{k!}$ h) $\frac{n!(n-2)!}{\left((n-1)!\right)^2}$

STEP 3: THE LABELING PRINCIPLE

In how many ways can we arrange the letters of the Swedish pop group name ABBA?

Answer: $\frac{4!}{2!2!} = \frac{24}{4} = 6$.

In how many ways can we arrange the letters of AABBBBA?

Answer: $\frac{7!}{3!4!}$

In how many ways can we arrange the letters of AAABBBBCCCCCC?

Answer: $\frac{13!}{3!4!6!}$

Let's look at this third problem and phrase it in a different way:

Mean Mr. Muckins has a class of 13 students. He has decided to call three of the students A students, four of them B students, and six of them C students. In how many ways could he assign these labels?

Answer: Let's imagine all thirteen are in a line.Here's one way he can assign labels:

A C B B B A C C C C A C B

Here's another way:

B A C C B C C C B A C B A

and so on.

We see that this labeling problem is just the same problem as rearranging letters. The answer must be $\frac{13!}{3!4!6!}$. □

Of 10 people in an office 4 are needed for a committee. How many ways?

Answer: Imagine the 10 people standing in a line. We need to give out labels. Four people will be called "ON" and six people will be called "LUCKY." Here is one way to assign those labels:

L L O O L O L L L O

We see that this is just a word arrangement problem. The answer is:

$$\frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210$$

□

In general, we have …

THE LABELING PRINCIPLE

Each of distinct N objects is to be given a label.

If a of them are to have label "1," b of them to have label "2," and so on, then the total number of ways to assign labels is:

$$\frac{N!}{a!b!\cdots z!}$$

PUTTING THE LABELING PRINCIPLE TO USE:

Three people from a group of twelve are needed for a committee. In how many different ways can a committee be formed?

Answer: The twelve folk are to be labeled as follows: 3 as "on the committee" and 9 as "lucky." The answer must be $\frac{12!}{3!\,9!} = 220$. □

COMMENT: Notice that we were sure to assign an appropriate label to <u>each and every</u> person (or object) in the problem. This fist the "self check" we described earlier.

Fifteen horses run a race. How many possibilities are there for first, second, and third place?

Answer: One horse will be labeled "first," one will be labeled "second," one "third," and twelve will be labeled "losers." The answer must be:
$\frac{15!}{1!\,1!1!12!} = 15 \times 14 \times 13 = 2730$. □

A "feel good" running race has 20 participants. Three will be deemed equal "first place winners," five will be deemed "equal second place winners," and the rest will be deemed "equal third place winners." How many different outcomes can occur?

<u>Answer</u>: Easy! $\frac{20!}{3!\,5!12!}$. □

From an office of 20 people, two committees are needed. The first committee shall have 7 members, one of which shall be the chair and 1 the treasurer. The second committee shall have 8 members. This committee will have 3 co-chairs and 2 co-secretaries and 1 treasurer. In how many ways can this be done?

Answer: Keep track of the labels. Here they are:

1 person will be labeled "chair of first committee"
1 person will be labeled "treasure of first committee"
5 people will be labeled "ordinary members of first committee"
3 people will be labeled "co-chairs of second committee"
2 people will be labeled "co-secretaries of second committee"
1 person will be labeled "treasurer of second committee"
2 people will be labeled "ordinary members of the second committee"
5 people will be labeled "lucky," they are on neither committee.

The total number of possibilities is thus: $\frac{20!}{1!1!5!3!2!1!2!5!}$. Piece of cake! □

Suppose 5 people are to be chosen from 12 and the order in which folk are chosen is not important. How many ways can this be done?

Answer: 5 people will be labeled "chosen" and 7 "not chosen. There are $\frac{12!}{5!7!}$ ways to accomplish this task. □

Suppose 5 people are to be chosen from 12 for a team and the order in which they are chosen is considered important. In how many ways can this be done?

Answer: We have:

1 person labeled "first"
1 person labeled "second"
1 person labeled "third"
1 person labeled "fourth"
1 person labeled "fifth"
7 people labeled "not chosen"

This can be done $\frac{12!}{1!1!1!1!1!7!}$ ways. □

STEP 4: MULTI-STAGELABELING

There are 7 men and 6 women in an office. How many ways are there to make a committee of five if …

a) Gender is irrelevant?
b) The committee must be all male?
c) The committee must consist of 2 men and 3 women?

Answer: a) This is just a problem of assigning labels to 13 people: five are "ON" and eight are "OFF":

$$\frac{13!}{5!8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 9 = 1287 \text{ ways}$$

b) This is a problem just among the men: 7 people to be labeled:

$$\frac{7!}{5!2!} = 21 \text{ ways}$$

c) **THIS IS A TWO STAGE PROBLEM!**

<u>Task 1</u>: Deal with the men

$$\frac{7!}{2!5!} = 21 \text{ ways}$$

<u>Rask 2</u>: Deal with the women

$$\frac{6!}{3!3!} = 20 \text{ ways}$$

By the multiplication principle, there are $21 \times 20 = 420$ ways to form the committee. □

Example: There are:

20 Americans
10 Australians (which include me, Dr. T.)
10 Austrians

Fourteen are needed for a math team. How many ways if …

a) Nationality is irrelevant?
b) The team must be all American?
c) The team must have 5 Americans, 5 Australians, and 4 Austrians?
d) Nationality is irrelevant, but Dr. T. must be on the team?

Answer:

a) $\frac{40!}{14!26!}$ (Do you see why?)

b) $\frac{20!}{14!6!}$ (Do you see why?)

c) Deal with Americans: $\frac{20!}{5!15!}$

Deal with the Australians: $\frac{10!}{5!5!}$

Deal with the Austrians: $\frac{10!}{4!6!}$

By the multiplication principle there are a total of $\frac{20!}{5!15!}\times\frac{10!}{5!5!}\times\frac{10!}{4!6!}$ ways to make a team.

d) With Dr. T. on the problem is now to select 13 more team members from 39 people. There are $\frac{39!}{13!26!}$ ways to do this.

□

Example: In how many ways can one arrange the letters AMERICA …

a) if there are no restrictions?
b) if the rearrangement must begin with M?
c) if the rearrangement must begin with M and end with C?
d) if the rearrangement must begin with A?

Answer: a) $\frac{7!}{2!}$ (Why?)

b) This is just a matter of arranging the letters AERICA in six slots:

There are $\frac{6!}{2!}$ ways.

c) This is a matter of arranging the letters AERIA in five slots:

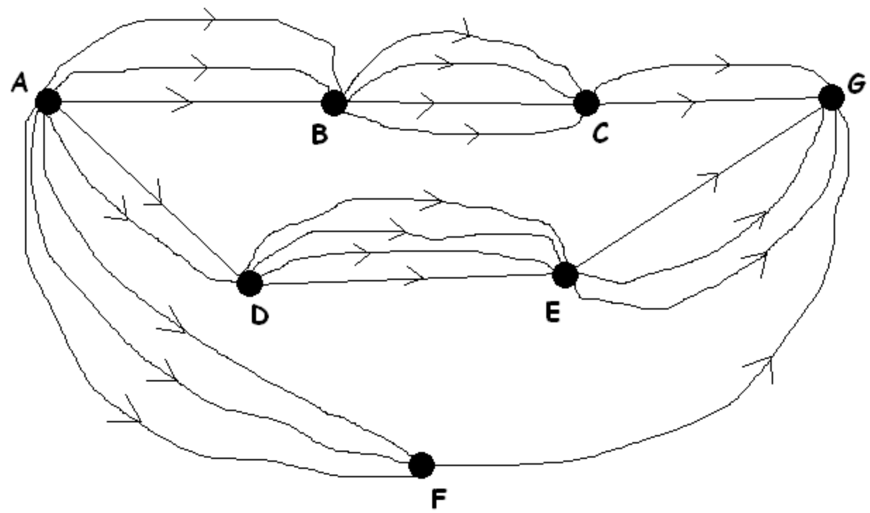
There are $\frac{5!}{2!}$ ways.

d) We need to arrange the letters MERICA in six slots:

There are $6!$ ways. □

EXERCISES:

Question 1: How many different paths are there from A to G?



Question 2:

a) In how many different ways can one arrange five As and five Bs.
b) A coin is tossed 10 times. In how many different ways could exactly five heads appear?

Question 3: The word BOOKKEEPING is the only word in the English language (ignoring its variants bookkeeper, etc.) with three consecutive double letters. In how many ways can one arrange the letters of this word?

Question 4: A multiple choice quiz has 10 questions each with 4 different possible answers. In how many ways can one fill out the quiz?

Question 5: In how many ways can one write down three vowels in order from left to right? How does the answer change if we insist that the vowels all be different?

Question 6: Ten people are up for election. In how many ways can one fill out a ballot for "president" and for "vice president"?

Question 7:

a) A mathematics department has 10 members. Four members are to be selected for a committee. In how many different ways can this be done?
b) A physics department has 10 members and a committee of four is needed. In that committee, one person is to be selected as "chair." In how many different ways can one form a committee of four with one chair?
c) An arts department has 10 members and a committee of four is needed. This committee requires two co-chairs. In how many different ways can one form a committee of four with two co-chairs?
d) An English department has 10 members and two committees are needed: One with four members with two co-chairs and one with three members and a single chair. In how many different ways can this be done?

Question 8: Three people from 10 will be asked to sit on a bench: one on the left end, one in the middle, and one on the right end. In how many different ways can this be done?

Question 9: Five pink marbles, two red marbles, and three rose marbles are to be arranged in a row. If marbles of the same colour are identical, in how many different ways can these marbles be arranged?

Question 10:

a) Hats are to be distributed to 20 people at a party. Five hats are red, five hats are blue, and 10 hats are purple. In how many different ways can this be done? (Assume the people are mingling and moving about.)
b) **CHALLENGE:** If the 20 people are clones and cannot be distinguished, in how many essentially different ways can these hats be distributed?

Question 11: In how many ways can one arrange the letters of NOODLEDOODLE if the arrangement must begin with an L and end with an E?

Question 12: A committee of five must be formed from five men and seven women.

a) How many committees can be formed if gender is irrelevant?
b) How many committees can be formed if there must be at exactly two women on the committee?
c) How many committees can be formed if one particular man must be on the committee and one particular woman must not be on the committee?
d) **CHALLENGE:** How many committees can be formed if one particular couple (one man and one woman) can't be on the committee together?

Question 13:

a) Twelve white dots lie in a row. Two are to be coloured red. In how many ways can this be done?
b) Consider the equation $10 = x + y + z$. How many solutions does it have if each variable is to be a positive integer or zero?

Question 14:

a) In how many ways can the letters ABCDEFGH be arranged?
a) In how many ways can the letters ABCDEFGH be arranged with letter G appearing somewhere to the left of letter D?
b) In how many ways can the letters ABCDEFGH be arranged with the letters F and H adjacent?

APPENDIX: FUN WITH POKER HANDS

One plays poker with a deck of 52 cards, which come in 4 suits (hearts, clubs, spades, diamonds) with 13 values per suit (A, 2, 3, …, 10, J, Q, K).

In poker one is dealt five cards and certain combinations of cards are deemed valuable. For example, a "four of a kind" consists of four cards of the same value and a fifth card of arbitrary value. A "full house" is a set of three cards of one value and two cards of a second value. A "flush" is a set of five cards of the same suit. The order in which one holds the cards in ones hand is immaterial.

EXAMPLE: How many flushes are possible in poker?

Answer: Again this is a multi-stage problem with each stage being its own separate labeling problem. One way to help tease apart stages is to image that you've been given the task of writing a computer program to create poker hands. How will you instruct the computer to create a flush?

First of all, there are four suits – hearts, spades, clubs and diamonds – and we need to choose one to use for our flush. That is, we need to label one suit as "used" and three suits as "not used." There are $\frac{4!}{1!3!} = 4$ ways to do this.

Second stage: Now that we have a suit, we need to choose five cards from the 13 cards of that suit to use for our hand. Again, this is a labeling problem - label five cards as "used" and eight cards as "not used." There are $\frac{13!}{5!8!} = 1287$ ways to do this.

By the multiplication principle there are $4 \times 1287 = 5148$ ways to compete both stages. That is, there are 5148 possible flushes. □

<u>Comment</u>: There are $\frac{52!}{5!47!} = 2598960$ five-card hands in total in poker.

(Why?) The chances of being dealt a flush are thus: $\frac{5148}{2598960} \approx 0.20\%$.

EXAMPLE: How many full houses are possible in poker?

<u>Answer</u>: This problem is really a three-stage labeling issue.

First we must select which of the thirteen card values – A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K - is going to be used for the triple, which will be used for the double, and which 11 values are going to be ignored. There are $\frac{13!}{1!1!11!} = 13 \times 12 = 156$ ways to accomplish this task.

Among the four cards of the value selected for the triple, three will be used for the triple and one will be ignored. There are $\frac{4!}{3!1!} = 4$ ways to accomplish this task. Among the four cards of the value selected for the double, two will be used and two will be ignored. There are $\frac{4!}{2!2!} = 6$ ways to accomplish this.

By the multiplication principle, there are $156 \times 4 \times 6 = 3744$ possible full houses. □

COMMENT: High-school teacher Sam Miskin recently used this labeling method to count poker hands with his high-school students. To count how many "one pair hands" (that is, hands with one pair of cards the same numerical value and three remaining cards each of different value) he found it instructive bring 13 students to the front of the room and hand each student four cards of one suit from a single deck of cards.

He then asked the remaining students to select which of the thirteen students should be the "pair" and which three should be the "singles." He had the remaining nine students return to their seats.

He then asked the "pair" student to raise his four cards in the air and asked the seated students to select which two of the four should be used for the pair. He then asked each of the three "single" students in turn to hold up their cards while the seated students selected on one the four cards to make a singleton.

This process made the multi-stage procedure clear to all and the count of possible one pair hands, namely,

$$\frac{13!}{1!3!9!} \times \frac{4!}{2!2!} \times 4 \times 4 \times 4$$

readily apparent.

EXERCISE: "Two pair" consists of two cards of one value, two cards of a different value, and a third card of a third value. What are the chances of being dealt two-pair in poker?

EXAMPLE: A "straight" consists of five cards with values forming a string of five consecutive values (with no "wrap around"). For example, 45678, A2345 and 10JQKA are considered straights, but KQA23 is not. (Suits are immaterial for straights.)

How many different straights are there in poker?

Answer: A straight can begin with A, 2, 3, 4, 5, 6, 7, 8, 9 or 10. We must first select which of these values is to be the start of our straight. There are 10 choices.

For the starting value we must select which of the four suits it will be. There are 4 choices.

There are also 4 choices for the suit of the second card in the straight, 4 for the third, 4 for the fourth, and 4 for the fifth.

By the multiplication principle, the total number of straights is:

$$10 \times 4 \times 4 \times 4 \times 4 \times 4 = 10240.$$

The chances of being dealt a straight is about 0.39%. □

SOLUTIONS

PAGE 3 SOLUTIONS:

1. For each route one chooses to travel from A to B, there are four options as to which route to take from B to C.

Top route A to B with four options B to C
PLUS
Middle route A to B with four options B to C
PLUS
Bottom route A to B and with four options B to C.

We have three groups of four, $4+4+4$, which corresponds to multiplication: 3×4.

2. For each of the twelve routes from A to B there are six options from C to D. We have $6+6+6+6+6+6+6+6+6+6+6+6=12\times 6$.

3. $5\times 4\times 2=40$

4. $10\times 10=100$

PAGE 4 SOLUTIONS:

5. The caveat in the statement is important. For example, in the clothing example, if I would never wear my mushroom-pink trousers with my lilac shirt, then the computation $5\times 4\times 2$ would not apply. In the movie example, if I only ever eat popcorn during a romance movie, then the computation 10×10 does not apply.

PAGE 5 SOLUTIONS:

6. $5\times 4\times 3\times 2\times 1=120$

7. $6\times 5\times 4\times 3\times 2\times 1=720$

PAGE 6 SOLUTIONS:

8. $11!$

PAGE 7 SOLUTIONS:

9. Three ways: BOB, BBO, OBB

10. No! Listing them doesn't seem fun!

PAGE 9 SOLUTIONS:

11. If the letters were distinguishable, there would be $14!$ ways to arrange them. However, this answer "collapses" by a factor of $6!$ if we remove distinguishing subscripts from the Es (there are $6!$ ways to arrange the Es amongst themselves in any particular pattern of letters). So we need to divide this answer by $6!$. But these answers will "collapse" by a factor of $3!$ if we remove subscripts from the Ss and again by a factor of $2!$ if we do the same for the Ts. So our initial answer $14!$ is divided be each of $6!$, $3!$ and $2!$.

PAGE 10 SOLUTIONS:

12.

$$\begin{aligned}\frac{16!}{5!6!} &= \frac{16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1 \quad\times\quad 6\times5\times4\times3\times2\times1} \\ &= \frac{16\times15\times14\times13\times12\times11\times10\times9\times8\times7}{5\times4\times3\times2\times1} \\ &= 16\times15\times14\times13\times11\times9\times8\times7 \\ &= 190270080\end{aligned}$$

13. Maybe. "Multiply together all the numbers from 1 up to 1." I guess this means $1!=1$ (and there isn't much multiplication to do!)

PAGE 11 SOLUTIONS:

14. a) 800 b) 105 c) $\frac{1}{88\times89}$

15. d) 1 e) N f) $n(n-1)$ g) $k+2$ h) $\frac{n}{n-1}$

EXERCISES PAGE 19 ONWARDS

Question 1: $3\times 4\times 2+2\times 4\times 3+3\times 1=51$ ways

Question 2: a) $\frac{10!}{5!5!}$

b) This is the same problem (with the same answer) but with the letters "H" and "T" instead.

Question 3: $\frac{11!}{2!2!2!}$

Question 4: This is a series of ten tasks to complete.

$4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4=4^{10}$ ways.

Question 5: This is a series of three tasks to complete.

If letters can be repeated: $5\times 5\times 5=125$ ways.

If letters cannot be repeated, $5\times 4\times 3=60$ ways.

Question 6: $\frac{10!}{1!1!8!}=10\times 9=90$

Question 7: Keep track of labels.

a) $\frac{10!}{4!6!}$ b) $\frac{10!}{1!3!6!}$ c) $\frac{10!}{2!2!6!}$ d) $\frac{10!}{2!2!1!2!3!}$

Question 8: Think: Sit left, sit middle, sit right, don't sit. $\frac{10!}{1!1!1!7!}$

Question 9: $\frac{10!}{5!2!3!}$

Question 10: a) $\frac{20!}{5!5!10!}$

b) ONE! If folk are identical then all distributions of hats look the same!

Question 11: $\frac{10!}{1!4!3!1!1!}$

Question 12: a) 12 people with five on and seven off: $\frac{12!}{5!7!} = 792$

b) We need to select three men and two women: $\frac{5!}{3!2!} \times \frac{7!}{2!5!} = 210$

c) We need to select four more people for the committee from a group of ten: $\frac{10!}{4!6!} = 210$.

d) There are $\frac{10!}{3!7!}$ committees with them on together. (We need three people from ten for the remainder of the committee.) As there are $\frac{12!}{5!7!}$ committees in total, there are $\frac{12!}{5!7!} - \frac{10!}{3!7!} = 792 - 120 = 672$ that avoid them both on together.

Question 13: a) $\frac{12!}{2!10!} = 66$ ways.

b) Do you see this is the same problem? Each picture of 12 dots with two coloured red gives three values for x, y and z: count the number of white dots to the left of the first red dot, the number of white dots between the two red dots, and the number of white dots to the right of the second red dot. And conversely, any solution to $10 = x + y + z$ corresponds to a picture of dots. The answer is also 66.

Question 14:

a) $8! = 40320$

b) Of the $8!$ arrangements in total half will have G to the left of D and half will have D to the left of G. There are thus $8!/2 = 20160$ arrangements of the type we seek.

c) Imagine the letters F and H stuck together as a single block, either as "FH" or as "HF". There are $7!$ ways to arrange A, B, C, D, E, G, and "FH"; and there are $7!$ ways to arrange A, B, C, D, E, G, and "HF". Thus there are $2 \times 7! = 10080$ arrangements with F and H adjacent.