

# Basic Set Theory

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LX 502 - Semantics I  
September 11, 2008

## 1. Motivation

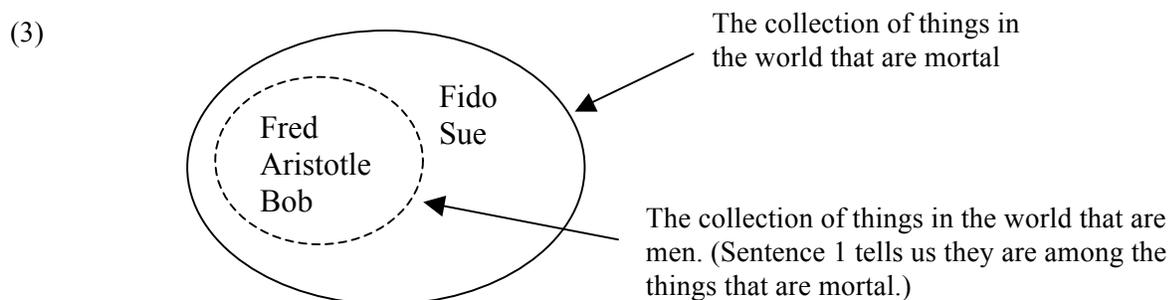
When you start reading these notes, the first thing you should be asking yourselves is “What is Set Theory and why is it relevant?” Though Propositional Logic will prove a useful tool to describe certain aspects of meaning, like the reasoning in (1), it is a blunt instrument.

- (1)    a. Jacques sang and danced  
          Therefore: Jacques sang
- b. If Jacques kisses Smurfette, she will slap Jacques  
          Jacques kissed Smurfette  
          Therefore: Smurfette slaps Jacques

Propositional Logic won't help us with the argument in (2), which is why we will extend it to Predicate Logic.

- (2)    Every man is mortal  
          Aristotle is a man  
          Therefore: Aristotle is mortal.

Our intuitions tell us that if the first and second sentence are true, then the third necessarily follows. The reasoning goes like this. Suppose the first sentence is true: the collection of men is part of the collection of mortal things. Now suppose the second sentence is also true: Aristotle is a man. The first sentence says that if a thing is in the collection of men, it is also in the collection of mortal things. So it follows that Aristotle is a mortal thing (or simply mortal), which is what the third sentence states. This is represented pictorially in (3).



The semantics of Predicate Logic is defined in terms of *Set Theory*.

## 2. Sets and Set Membership

A bunch of carrots, a classroom full of students, a herd of elephants: these are all examples of sets of things. Intuitively, you can think of a set as an abstract collection of objects, which may correspond to things in the world or to concepts, etc. A set is sometimes also called a *collection*.<sup>1</sup> The objects that are collected in a set are called its *members or elements*. So, a carrot, a student, and an elephant, are members of the sets described above. A Set may also contain other sets as its members.

I offer no definition of what a set is beyond the intuitive notion described above. Instead, I am going to show you what can be done with sets. This is a typical approach to Set Theory, i.e., sets are treated as primitive s of the theory and are not definable in more basic terms. I adopt the notation in (4) for convenience.

- (4)    a. Capital letters represent sets: A, B, C, ...  
      b. Lower case letters represent members of sets: a, b, c, ..., sometimes x, y, z, ...

The principal concept of Set Theory is *belonging*, i.e., elements belonging to sets. And this concept is represented using the membership relation, expressed by the rounded Greek letter epsilon  $\in$ , as in (5). It's negation is expressed using a barred epsilon  $\notin$  as in (6).

- (5)     $x \in A$   
      *x is an element of the set A*

- (6)     $x \notin A$   
      *x is not an element of the set A*

I noted above that sets can be elements of sets. For example, a municipality is a set of people; a city a set of municipalities; a state a set of cities; a country a set of states; and finally, the UN a set of countries. The set representing the UN has, as its members, various countries. Each country has various states as its member.

## 3. Specifying a Set

### 3.1. List Notation

The conventional way of specifying a set is to list its members between curly braces and to separate each member with a comma. This is shown in (7).

- (7)    {Ginny, Ron, Fred, George, Percy, Charlie, Bill}  
      {a, e, i, o, u}  
      {52, ♣, Bob, +}

Although this method of representation suggests that the members have an order to them, this is not the case. The members in a set have no particular order. The order that the elements are listed in is irrelevant. For example, the set in (8a) is equivalent to the set in (8b).

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<sup>1</sup> In mathematics, the terms *group* and *class* are reserved for other mathematical objects. So I will avoid these terms.



(13)  $\{ x \mid x \text{ is a natural number} \}$

*The set of all  $x$  such that  $x$  is a natural number*

The curly braces tell us this is a set. The letter  $x$  is a **variable** that stands in for **any** object that meets the criteria described after the vertical line. The vertical line serves as a divider and it is spoken “such that”. What follows the line is a condition that each object must satisfy to be included in the set. The set in (13) is equivalent to the set in (14).

(14)  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$

The same set may be described in several different ways, illustrated in (15). The manner in which a set is defined is irrelevant. Its membership is what is important.

- (15) a.  $\{ y \mid y \text{ is a natural number between 1 and 9 that is divisible by 2} \}$   
b.  $\{ z \mid z \text{ is an even number between 1 and 9} \}$   
c.  $\{ 2, 4, 6, 8 \}$

It turns out that the predicate notation is actually too powerful and generates paradoxes, like the one in (16). The sets we deal with are well-behaved so this won't effect us so much, but it is still something to keep in mind and it forces us to introduce an additional component in the predicate notation.

(16) **Russell's Paradox**

Let  $X = \{ y \mid y \notin y \}$ . Is the set  $X$  a member of itself?

Assume  $X \in X$ . This implies that  $X \notin X$ , which contradicts our assumption

Assume  $X \notin X$ . This implies that  $X \in X$ , which contradicts our assumption... paradox!

To avoid this paradox (and similar ones), we need to restrict the predicate notation. The problem is that the variable  $x$  in (16) admits any kind of object, even really deviant ones. We need to restrict the variable  $x$  to well-behaved objects. We do this by requiring that sets defined in this way are “carved out” of bigger sets. An example is provided in (17).

(17)  $B = \{ x \in U \mid x \text{ is an even number} \}$

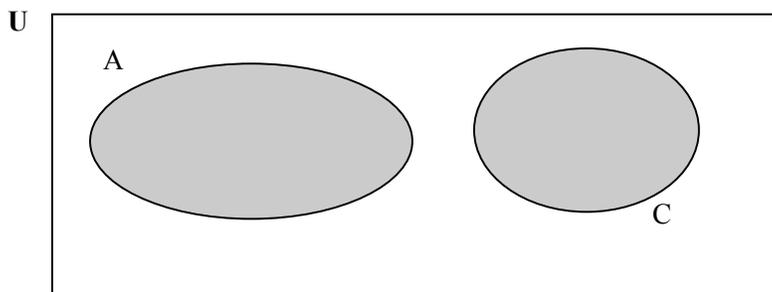
*The set of all  $x$  in  $U$  such that  $x$  is an even number*

The set  $U$  is called the *Universe*. The set  $B$  in (17) is formed from the elements that are already in  $U$ . That's what I mean when I say “ $B$  is carved out of  $U$ ”. The examples in (18) illustrate this.

- (18) a. Let  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$  and  $B = \{ x \in U \mid x \text{ is an even number} \}$   
Then  $B = \{2, 4, 6, 8, 10\}$ , i.e.,  $B$  contains those elements from  $U$  that are even  
b. Let  $U = \{1, 2, 3, 4, 10, 12, 13, 15, 20\}$  and  $B = \{ x \in U \mid x \text{ is an even number} \}$   
Then  $B = \{ 2, 4, 10, 12, 20 \}$ , i.e.,  $B$  contains those elements from  $U$  that are even

Using a Venn Diagram, the universe  $U$  is depicted as a box that contains every circle. The diagram in (19), for instance, tells us that the sets  $A$  and  $C$  are carved out of the set  $U$  and have no elements in common. These notions will be made more precise in the next section.

(19) The Universe



## 4. Set-Theoretic Relations

### 4.1. Equality

Now that we have a grasp of what sets are, we can define two important relations between sets. The first is equality. We need to define what it means for two sets to be equal. This is usually stated as an Axiom (called the *Axiom of Extension*). For two sets,  $A$  and  $B$ , if they satisfy (20), we write  $A = B$ ; otherwise  $A \neq B$ .

(20) Two sets are equal if and only if they have the same elements

I want to convince you that (20) is not a trivial statement. Let me show you where it fails. Suppose  $x$  and  $A$  are human beings and that  $x \in A$  means  $x$  is an ancestor of  $A$ . (By ancestor, I mean parents, parents' parents, parents' parents' parents, etc.) Consider the two sets  $A$  and  $B$ . Suppose I want to determine if they are the same person in a genealogical study. The definition of equality in (20) says that  $A = B$  if and only if  $A$  and  $B$  have the same elements. In other words, in my example, if  $A$  and  $B$  are the same person, then they have the same ancestors. This is clearly true. However, (20) also says that if  $A$  and  $B$  have the same elements, then  $A = B$ . In other words, if  $A$  and  $B$  have the same ancestors, they are the same person. This is false. My brother and I have the same ancestors but we are not the same person.

### 4.2. Subsets

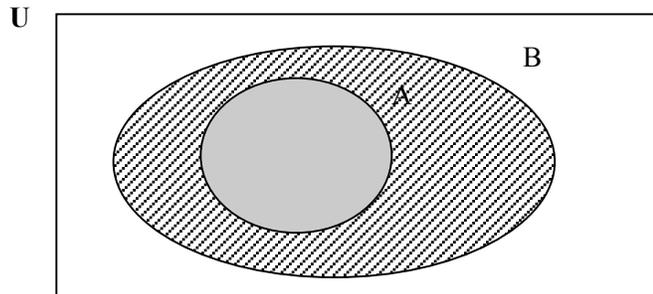
The second relation I want to define is called the *subset relation*, stated in (21). This is an incredibly important relation that is closely related to the notion of entailment. For two sets,  $A$  and  $B$ , if they satisfy (21), then we write  $A \subseteq B$ .

(21)  $A$  is a subset of  $B$  if and only if every element of  $A$  is an element of  $B$

If  $A \subseteq B$ , we can also say “ $B$  includes  $A$ ”. The relation in (21) requires that every element of  $A$  is an element of  $B$ , but it says nothing about  $B$ . In fact,  $B$  can contain elements not in  $A$ . The Venn diagram in (22)

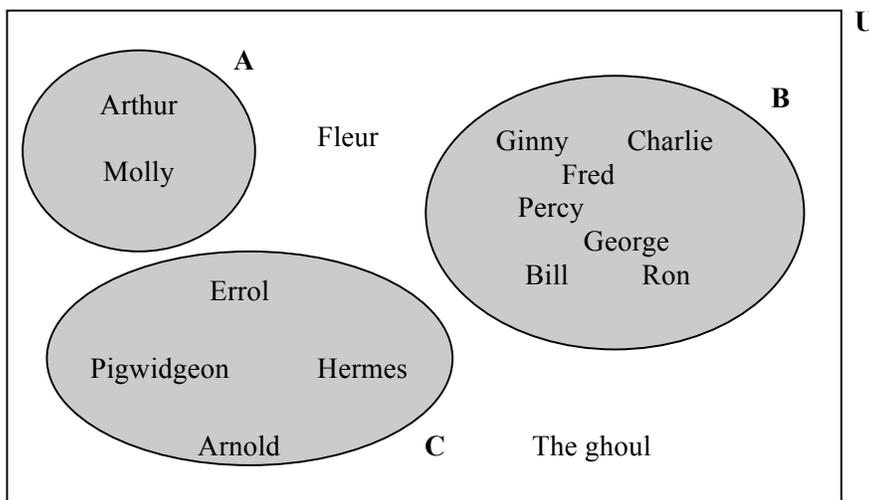
illustrates the relationship between A and B. This relationship also exists between the U and B and U and A. In other words,  $A \subseteq U$  and  $B \subseteq U$ .

(22) Subset Relation



Before moving on let me present a comprehensive example to illustrate things so far. This example is based on characters in the Harry Potter books.

(23) Example



$U = \{ \text{Arthur, Molly, Fleur, Ginny, Charlie, Fred, Percy, George, Bill, Ron, Errol, Pigwidgeon, Hermes, Arnold, the ghoul} \}$

$A = \{ x \in U \mid x \text{ is a Weasley parent} \}$   
 $= \{ \text{Arthur, Molly} \}$

$B = \{ x \in U \mid x \text{ is a Weasley child} \}$   
 $= \{ \text{Ginny, Ron, Charlie, Fred, George, Bill, Percy} \}$

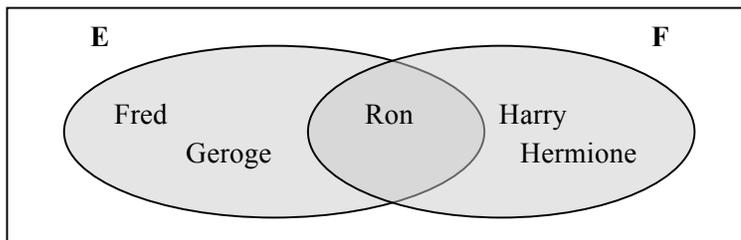
$C = \{ x \in U \mid x \text{ is a Weasley pet} \}$   
 $= \{ \text{Errol, Pigwidgeon, Arnold, Hermes} \}$

### 4.3. Union

There are three operations that take two sets and return a new set. The first is called *union* and, as the name suggests, it takes two sets and unites the elements into a comprehensive set. This operation is defined in (24) with an example in (25).

(24) The union of the set A and the set B is the set that contains all the elements that belong to A or to B, written  $A \cup B$ .

(25) Let  $E = \{\text{Ron, Fred, George}\}$  and let  $F = \{\text{Ron, Harry, Hermione}\}$   
Then  $E \cup F = \{\text{Ron, Fred, George, Harry, Hermione}\}$



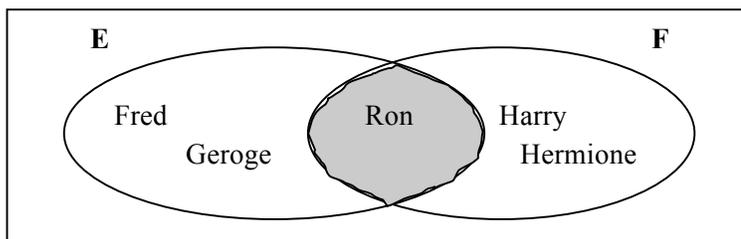
The shaded area represents  $E \cup F$

### 4.4. Intersection

The second operation is called *intersection*, defined in (26) with an example in (27).

(26) The intersection of the set A and the set B is the set that contains all the elements that belong to both A and B, written  $A \cap B$ .

(27) Let  $E = \{\text{Ron, Fred, George}\}$  and let  $F = \{\text{Ron, Harry, Hermione}\}$   
Then  $E \cap F = \{\text{Ron}\}$



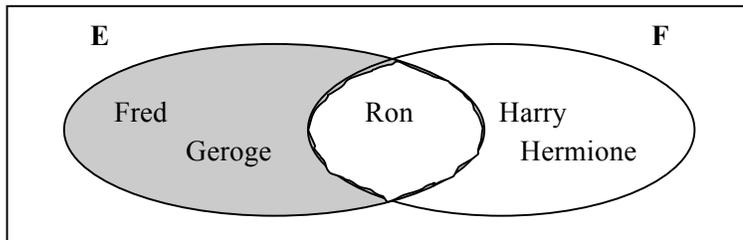
The shaded area represents  $E \cap F$

### 4.5. Complementation

The final operation is *set-theoretic difference*. The idea behind this operation is that it subtracts the elements of one set from another.

(28) The difference between the set A and the set B is the set that contains all the elements that belong to A but not to B

- (29) Let  $E = \{\text{Ron, Fred, George}\}$  and let  $F = \{\text{Ron, Harry, Hermione}\}$   
 Then  $E - F = \{\text{Fred, George}\}$



The shaded area represents  $E - F$

The difference  $U - A$  has a special name. This set is called the *complement of A*. It contains those elements in the universe set  $U$  that are not in  $A$ . It is written  $A'$  (or  $\bar{A}$ ).

- (30) Let  $U = \{\text{Ron, Fred, George, Harry, Hermione, Lupin, Sirius}\}$   
 $E = \{\text{Ron, Fred, George}\}$

Then  $E' = U - E = \{\text{Harry, Hermione, Lupin, Sirius}\}$

## 4.6 Summary

The formal definitions for union, intersection, difference, and complement are presented in (31).

- (31) For any two sets  $A$  and  $B$ ,
- $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
  - $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
  - $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$
  - $A' = \{x \in U \mid x \notin A\}$

## 5. The Empty Set

What about a set with no elements? The traditional way of defining this set is given in (32). It turns out there is only one set with no elements; it is called the *empty set* or *null set* and written with the special symbol  $\emptyset$ .

- (32)  $\{x \in U \mid x \neq x\}$

The empty set  $\emptyset$  has an interesting property that I argued for in class. The empty set is a subset of every set. The argument goes like this.  $A \subseteq B$  if and only if every element of  $A$  is an element of  $B$ . Pay close attention to the wording in the definition. The condition for subset-hood is that “every element of  $A$  is an element of  $B$ ”. Strictly speaking, the condition does not require that  $A$  have any elements. It just requires that if an element is

in A (if  $x \in A$ ) then that element is also in B (then  $x \in B$ ). If A has no elements, the condition is satisfied automatically.

I won't lie to you. This kind of reasoning is very abstract. Here's a way of thinking about it that might help. Imagine you're in Junior High School. Your parents are strict about homework and only let you go out when it is finished. Let A be the set of assignments you have and let B be the set of finished things. That means you can go out only if  $A \subseteq B$ . (Think of A as the homework-pile and B as the done-pile. Your parents want the homework-pile to be part of the done-pile.) Now suppose today your teacher never assigned homework. In other words,  $A = \emptyset$ . Are you allowed to go out? Yes. Why? The condition on you going out is that "every assignment you have is finished" not that you actually have homework.

## 6. Ordered Pairs

An **ordered pair** is a sequence of two elements,  $\alpha$  and  $\beta$ . We write these elements between angled brackets,  $\langle \alpha, \beta \rangle$ . This notation represents the fact that the first element in the sequence is  $\alpha$  and the second is  $\beta$ . You can think of an ordered pair as a pair of coordinates, like the coordinates on a map:  $\alpha$  is the first coordinate and  $\beta$  is the second. There are crucial differences between ordered pairs and sets. In an ordered pair, the order of the elements matters. So if the elements  $\alpha$  and  $\beta$  are different, then  $\langle \alpha, \beta \rangle \neq \langle \beta, \alpha \rangle$ . Furthermore, even if the elements are the same, each element is treated as a distinct coordinate of the ordered pair, i.e.,  $\langle \alpha, \alpha \rangle \neq \langle \alpha \rangle$ .

- (33) Ordered Pairs
- a.  $\langle a, b \rangle \neq \langle b, a \rangle$
  - b.  $\langle b, b \rangle \neq \langle b \rangle$

- (34) Sets
- a.  $\{a, b\} = \{b, a\}$
  - b.  $\{b, b\} = \{b\}$

Suppose we have two sets A and B and we form ordered pairs by taking an element of A as the first member of the pair and an element of B as the second member. The set containing all such ordered pairs is called the **Cartesian Product of A and B**, written  $A \times B$ . This set is defined using predicate notation in (35).

- (35) Cartesian Product of A and B
- $$A \times B = \{ \langle \alpha, \beta \rangle \mid \alpha \in A \text{ and } \beta \in B \}$$

The following are examples of the Cartesian Product.

- (36) Let  $K = \{a, b, c\}$  and  $L = \{1, 2\}$
- $$K \times L = \{ \langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle \}$$
- $$L \times K = \{ \langle 1, a \rangle, \langle 1, b \rangle, \langle 1, c \rangle, \langle 2, a \rangle, \langle 2, b \rangle, \langle 2, c \rangle \}$$
- $$K \times K = \{ \langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle, \langle b, a \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, a \rangle, \langle c, b \rangle, \langle c, c \rangle \}$$
- $$L \times L = \{ \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$$

(37) Let  $U = \{ \text{beard}, \text{sideburns}, \text{mohawk}, \text{dog} \}$

$$U \times U = \{ \langle \text{beard}, \text{beard} \rangle, \langle \text{beard}, \text{sideburns} \rangle, \langle \text{beard}, \text{mohawk} \rangle, \langle \text{beard}, \text{dog} \rangle, \langle \text{sideburns}, \text{beard} \rangle, \langle \text{sideburns}, \text{sideburns} \rangle, \langle \text{sideburns}, \text{mohawk} \rangle, \langle \text{sideburns}, \text{dog} \rangle, \langle \text{mohawk}, \text{beard} \rangle, \langle \text{mohawk}, \text{sideburns} \rangle, \langle \text{mohawk}, \text{mohawk} \rangle, \langle \text{mohawk}, \text{dog} \rangle, \langle \text{dog}, \text{beard} \rangle, \langle \text{dog}, \text{sideburns} \rangle, \langle \text{dog}, \text{mohawk} \rangle, \langle \text{dog}, \text{dog} \rangle \}$$

What about ordered **triples**? The notion of an ordered pair can be extended to an ordered triple by using the definitions in (38).

(38)  $\langle \alpha, \beta, \gamma \rangle =_{\text{def}} \langle \langle \alpha, \beta \rangle, \gamma \rangle$   
 $A \times B \times C =_{\text{def}} (A \times B) \times C$

These definitions simply tell us how to define three coordinates using our pre-existing definition of an ordered pair—just form a pair where the first coordinate is itself an ordered pair. The same goes for the Cartesian Product. I first form the Cartesian Product of A and B. The result is the set  $A \times B$ . I then form the Cartesian Product of  $(A \times B)$  and C. The result is the set  $(A \times B) \times C$ .

(39) Let  $L = \{1,2\}$

a.  $(L \times L) \times L = \{ \langle \alpha, \beta \rangle \mid \alpha \in (L \times L) \text{ and } \beta \in L \}$   
 $= \{ \langle \langle 1,1 \rangle, 1 \rangle, \langle \langle 1,1 \rangle, 2 \rangle, \langle \langle 1,2 \rangle, 1 \rangle, \langle \langle 1,2 \rangle, 2 \rangle, \langle \langle 2,1 \rangle, 1 \rangle, \langle \langle 2,1 \rangle, 2 \rangle, \langle \langle 2,2 \rangle, 1 \rangle, \langle \langle 2,2 \rangle, 2 \rangle \}$

b.  $L \times L \times L = \{ \langle 1,1,1 \rangle, \langle 1,1,2 \rangle, \langle 1,2,1 \rangle, \langle 1,2,2 \rangle, \langle 2,1,1 \rangle, \langle 2,1,2 \rangle, \langle 2,2,1 \rangle, \langle 2,2,2 \rangle \}$   
 $= \{ \langle \alpha, \beta, \gamma \rangle \mid \alpha \in L \text{ and } \beta \in L \text{ and } \gamma \in L \}$

We can do the same for ordered quadruples, written  $\langle \alpha, \beta, \gamma, \delta \rangle$ . And so forth. We won't go beyond triples in this course.

## 7. Relations and Functions

In natural language, **relations** describe a kind of connection existing between objects in the world. If it helps, think of a relation as a relationship between objects in the world. Some examples include: *is the mother of*, *is the neighbour of*, *is a part of*, *is older than*, *is an ancestor of*, *is a subset of*, *is dating*, etc. Formally a relation is defined as a set of ordered pairs. We sometimes write  $Rab$  or  $aRb$  to mean “*a* bears the relation *R* to *b*”, but officially we write  $\langle a,b \rangle \in R$ . When a relation is between two objects, we call it a binary relation.

(40) If A and B are sets and  $R \subseteq A \times B$ , we call R a **binary relation between A and B**.  
 A relation  $R \subseteq A \times A$  is called a **binary relation on A**.

(41) Example

Let  $R$  be the binary relation “is older than” on the set of Weasley boys<sup>2</sup>

$$R_{\text{is older than}} = \{ \langle \text{Fred}, \text{Ron} \rangle, \langle \text{George}, \text{Ron} \rangle, \langle \text{Percy}, \text{Ron} \rangle, \langle \text{Percy}, \text{Fred} \rangle, \langle \text{Percy}, \text{George} \rangle, \langle \text{Charlie}, \text{Ron} \rangle, \langle \text{Charlie}, \text{Fred} \rangle, \langle \text{Charlie}, \text{George} \rangle, \langle \text{Charlie}, \text{Fred} \rangle, \langle \text{Charlie}, \text{Percy} \rangle, \langle \text{Bill}, \text{Ron} \rangle, \langle \text{Bill}, \text{Fred} \rangle, \langle \text{Bill}, \text{George} \rangle, \langle \text{Bill}, \text{Fred} \rangle, \langle \text{Bill}, \text{Percy} \rangle, \langle \text{Bill}, \text{Charlie} \rangle \}$$

Interestingly, from this relation, we can conclude that Fred and George are the same age. Do you see how?  $\langle \text{George}, \text{Fred} \rangle \notin R$  and  $\langle \text{Fred}, \text{George} \rangle \notin R$ . Since neither is older than the other, it follows that they are the same age. We can also conclude that Ron is the youngest and Bill is the oldest. The former follows because  $\langle \text{Ron}, y \rangle \notin R$ , for any  $y$ . So Ron is not older than any other Weasley boy. The latter follows because  $\langle x, \text{Charlie} \rangle \notin R$ , for any  $x$ . So no other Weasley boy is older than Charlie.

To draw these conclusions, I looked at the first and second coordinate of the relation. It is sometimes helpful to isolate the first coordinates and second coordinates of a relation. What we do is look at each ordered pair in the relation, form a set from all the first coordinates (called **the domain of  $R$** ) and form a set from all the second coordinates (called **the range of  $R$** ).

(42) The **domain** and **range** of a relation  $R$  are the sets

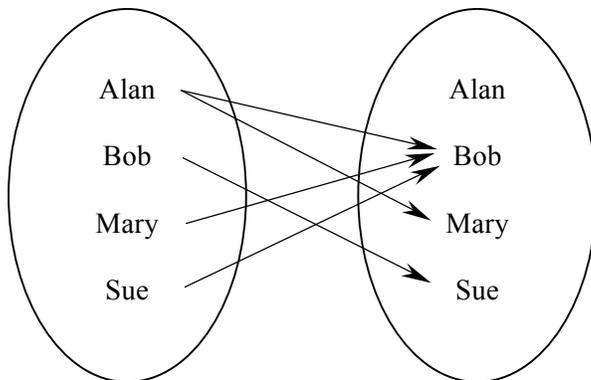
- i.  $\text{Domain}(R) = \{ \alpha \mid \langle \alpha, \beta \rangle \in R \text{ for some } \beta \}$
- ii.  $\text{Range}(R) = \{ \beta \mid \langle \alpha, \beta \rangle \in R \text{ for some } \alpha \}$

(43)  $\text{Domain}(R_{\text{is older than}}) = \{ \text{Fred}, \text{George}, \text{Percy}, \text{Bill}, \text{Charlie} \}$   
 $\text{Range}(R_{\text{is older than}}) = \{ \text{Ron}, \text{Fred}, \text{George}, \text{Percy}, \text{Bill} \}$

A relation can be represented visually in a diagram. Arrows are used to associate the elements of the two sets (possibly the same set). This is sometimes called a *mapping* or an *association*.

(44) Example

$R$  is the binary relation on  $A$ , *kissed*



<sup>2</sup> The Weasley boys form the set  $A = \{ \text{Ron}, \text{Fred}, \text{George}, \text{Percy}, \text{Charlie}, \text{Bill} \}$

A **function** is a special type of relation. It is a relation with a special restriction on the first coordinate. For a relation to be a function, it must be the case that every element in its domain is associated to one and only one element in its range.

(45) A set  $F \subseteq A \times B$  is a **function** if and only if  
for every  $x \in \text{Domain}(F)$ , there is one and only one  $y \in \text{Range}(F)$ , such that  $\langle x, y \rangle \in F$

This may sound complicated but it isn't. It simply says that, for a function, the first coordinate can only have one second coordinate. The relation "x's biological father is y" constitutes a function, since every person x has one and only one biological father y. The *kiss* relation represented in (44) is not a function because Alan has kissed two people:  $\langle \text{Alan}, \text{Bob} \rangle \in R_{\text{kiss}}$  and  $\langle \text{Alan}, \text{Mary} \rangle \in R_{\text{kiss}}$ . Alan has not kissed a unique person, or, more precisely, the first coordinate *Alan* is associated to more than one second coordinate, namely *Bob* and *Mary*. It should be equally clear that the relation *is taller than* for the Weasley boys in (41) is not a function. Percy, for example, is taller than three others: Fred, George, and Ron. In other words,  $\langle \text{Percy}, \text{Ron} \rangle \in R_{\text{is taller than}}$ ,  $\langle \text{Percy}, \text{Fred} \rangle \in R_{\text{is taller than}}$ , and  $\langle \text{Percy}, \text{George} \rangle \in R_{\text{is taller than}}$ .

Functions are typically written as lower case letters, e.g., *f* and *g*. They are represented using the conventional notation you learned in high school math,  $f(x) = y$ , rather than set notation. But you should keep in mind that this is equivalent to  $\langle x, y \rangle \in f$ .

(46)  $f_{x's \text{ biological father is}}(\text{Ron}) = \text{Arthur}$   
 $\langle \text{Ron}, \text{Arthur} \rangle \in f_{x's \text{ biological father is}}$