Summation

1. *Notation*

Many problems in mathematics and its applications (e.g., statistics) involve sums with more than two terms. Writing all of the terms in such a sum can be cumbersome, especially if there are many terms. In some cases we can use *ellipses*, for example we might write

$$1 + 2 + 3 + \cdots + 100$$

to indicate the sum of the integers from one to one hundred. But this is a little vague, and in many cases, it might not be clear what terms are missing.[†]

The Swiss mathematician Leonard Euler (pronounced *oiler*) introduced notation for sums, using the greek letter Σ, which is an upper-case *sigma*.[‡]

Definition: *Given the terms,* $A_m, A_{m+1}, A_{m+2}, \ldots, A_n$, *we denote their sum by*

$$A_m + A_{m+1} + A_{m+2} + \cdots + A_n = \sum_{k=m}^{n} A_k. \tag{1.1}$$

The variable m *is called the* **lower limit of summation**, n *is called the* **upper limit of summation** *and* k *is called the* **index of summation**.

In this notation it is understood that m and n are integers and $m \leq n$. Furthermore, the index of summation k increases by increments of 1, starting from m and ending at n.

The terms $\{A_m, \ldots, A_n\}$ may be a list of numbers (e.g., data from an experiment), in which case the index k simply enumerates the list. On the other hand, the terms A_k may depend on k, i.e., A_k may be a function of k — this is the case that we will be considering here.

In all of the following examples, the terms of the sum are explicit functions of the index of summation.

Examples.

1a. $\displaystyle\sum_{k=1}^{6} k = 1 + 2 + 3 + 4 + 5 + 6.$

1b. $\displaystyle\sum_{j=0}^{100} (2j) = 0 + 2 + 4 + \cdots + 200.$

[†]Can you tell what terms are missing from the sum $1 + 5 + 11 + \cdots + 109$?

[‡]This notation for sums is therefore also called 'sigma notation'.

1c. $\sum_{i=3}^{15}(i^2+i-1) = 11+19+29+\cdots+239.$

Comments: As you can see in the examples, we can use letters other than k to denote the index of summation. The letters i, j and k are common choices, as are m and n (when they're not being used to denote the limits of summation). Also, as you can see in example **1b.**, the lower limit of summation may be zero (or negative).

Sigma notation is simply that — notation. It is not a tool for *evaluating* the sums in question. To evaluate sums, we'll use the basic properties of addition to develop some simple rules and formulas. On the other hand, the Σ-notation will make these rules and formulas easier to express and understand.

2. *Basic rules.*

The only operation being used in the sum $\sum_{k=m}^{n} A_k$ is addition. It follows that all the basic properties of addition hold for such sums. In particular, we can rearrange the terms in a sum, we can collect terms to split a sum into smaller sums and multiplication by a constant factor *distributes* over a sum. The following three rules illustrate these properties.

R1. Rearranging terms: $\sum_{k=m}^{n}(A_k+B_k) = \left(\sum_{k=m}^{n} A_k\right) + \left(\sum_{k=m}^{n} B_k\right).$

R2. Collecting terms: $\sum_{k=m}^{n} A_k = \sum_{k=m}^{l} A_k + \sum_{k=l+1}^{n} A_k.$

R3. Distributive property: $\sum_{k=m}^{n} cA_k = c\left(\sum_{k=m}^{n} A_k\right).$

Examples.

2a. $\sum_{k=1}^{6} k = \sum_{k=1}^{3} k + \sum_{k=4}^{6} k.$

2b. $\sum_{j=0}^{100}(2j) = 2\left(\sum_{j=0}^{100} j\right).$

2c. $\sum_{i=3}^{15}(i^2+i-1) = \sum_{i=3}^{15} i^2 + \sum_{i=3}^{15} i + \sum_{i=3}^{15}(-1).$

Comments: Rules **1.** and **2.** rely on the *associative* and *commutative* properties of addition. Also, as you can see in the last sum on the right of Example **2c.**, the terms in a sum may also be *constant*.

3. *Simple formulas.*

One way of evaluating a sum is to simply add all of the terms together, one after another, until we're done. This is easy to do when the number of terms is small, e.g.,

$$\sum_{k=1}^{6} k = 1+2+3+4+5+6 = 3+3+4+5+6 = 6+4+5+6 = 10+5+6 = 15+6 = 21,$$

and in many cases this may be the only option (for example in statistical applications). On the other hand, there are many sums that can be evaluated using formulas. In this section, I'll introduce two simple formulas, and in the next two sections we'll get more sophisticated.

In all of these formulas I'll consider sums from 1 to n,[§] and the values of the sums we consider will all be functions of the upper limit, n.

We begin with the formula for a sum of constant terms. This formula is self-explanatory:

$$\sum_{k=1}^{n} c = \overbrace{c + c + c + \cdots + c}^{n} = n \cdot c. \tag{3.1}$$

Next, we consider the sum of the first n integers: $1 + 2 + 3 + \cdots + n$. According to mathematical legend, the famous mathematician Karl Friedrich Gauss discovered this formula when he was about seven years old using the argument below.

Write $S_{n,1} = \sum_{k=1}^{n} k$. To find a formula for $S_{n,1}$, we compute the value of $2S_{n,1}$ and divide the result by 2. To compute $2S_{n,1}$, we add $S_{n,1}$ to itself in a clever way — we write the terms in $S_{n,1}$ once in the usual (increasing) order, and below the first sum, we write the same terms in reverse (decreasing) order, then we add the terms column by column before adding the results of the columns:

$$\begin{array}{rcccccccccccc}
S_{n,1} & = & & 1 & + & 2 & + & 3 & + & \cdots & + & n & \\
+S_{n,1} & = & + & n & + & n-1 & + & n-2 & + & \cdots & + & 1 & \\
\hline
2S_{n,1} & = & & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & = n(n+1).
\end{array}$$

This gives the formula:

$$S_{n,1} = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{3.2}$$

Examples.

3a. $\displaystyle\sum_{i=1}^{10} i = \frac{10 \cdot 11}{2} = 55.$

[§] To apply these formulas to sums whose lower limit of summation is different from 1, we use rule **R2.**. See example **3c.**

3b. $\displaystyle\sum_{j=1}^{20}(3j+2) = \sum_{j=1}^{20}(3j) + \sum_{j=1}^{20} 2 = 3\sum_{j=1}^{20} j + 20 \cdot 2 = 3 \cdot \frac{20 \cdot 21}{2} + 40 = 670.$

Note how I used properties **R1.** and **R3.**, before applying formulas (3.1) and (3.2).

3c. $\displaystyle\sum_{k=10}^{85} \frac{k}{3} = \frac{1}{3}\sum_{k=10}^{85} k = \frac{1}{3}\left(\sum_{k=1}^{85} k - \sum_{k=1}^{9} k\right) = \frac{1}{3}\left(\frac{85 \cdot 86}{2} - \frac{9 \cdot 10}{2}\right) = \frac{3610}{3}.$

In this example, I first used property **R2.** to express the original sum from 10 to 85 as the difference of the sums from 1 to 85 and the sum from 1 to 9,

$$\sum_{k=1}^{85} k = \sum_{k=1}^{9} k + \sum_{k=10}^{85} k \implies \sum_{k=10}^{85} k = \sum_{k=1}^{85} k - \sum_{k=1}^{9} k,$$

and then I applied formula (3.2).

4. *Telescoping sums.*

In certain important cases, most of the terms of a sum cancel each other out, leaving only one or two terms. A sum like this is often called a *telescoping sum*, because it collapses like a telescope. We can take advantage of this phenomenon to find more formulas for sums.

Consider, for example, the sum

$$\sum_{k=1}^{n}(2k+1). \tag{4.1}$$

At first glance, there does not appear to be any cancelation, and indeed there isn't. On the other hand, by using a familiar identity from high school, we can express each term in such a way, that that the resulting sum collapses down to two terms. The identity I speak of is $(k+1)^2 = k^2 + 2k + 1$, which allows us to write

$$2k + 1 = (k+1)^2 - k^2. \tag{4.2}$$

If we replace the terms in the sum in (4.1) by the right-hand side of (4.2), we obtain

$$\begin{aligned} \sum_{k=1}^{n}(2k+1) &= \sum_{k=1}^{n}(k+1)^2 - k^2 \\ &= (2^2 - 1^2) + (3^2 - 2^2) + (4^2 - 3^2) + \cdots + (n^2 - (n-1)^2) + ((n+1)^2 - n^2) \\ &= (n+1)^2 - 1^2. \end{aligned}$$

In other words, we have

$$\sum_{k=1}^{n}(2k+1) = n^2 + 2n, \tag{4.3}$$

a formula that will guide us in the next section.

We can also use telescoping sums to find the sum of a *geometric sequence*. Recall that a sequence of the form $q,\ q^2,\ q^3,\ \ldots\ , q^n$ is called a geometric sequence. Our goal is to find the value of the sum

$$q + q^2 + \cdots + q^n = \sum_{k=1}^{n} q^k.$$

This sum is not telescoping, but we can transform it into a telescoping sum by using the distributive property, **R3.**, as follows. First we multiply the sum above by $(q-1)$, and then distribute this factor to obtain a telescoping sum:

$$(q-1)\sum_{k=1}^{n} q^k = \sum_{k=1}^{n}(q-1)q^k = \sum_{k=1}^{n}\left(q^{k+1} - q^k\right) = q^{n+1} - q.$$

Dividing both sides of this last equation by $(q-1)$ yields the formula

$$\sum_{k=1}^{n} q^k = \frac{q^{n+1} - q}{q-1}, \tag{4.4}$$

which is valid for any number $q \neq 1$.[¶]

Examples.

4a. $\displaystyle\sum_{m=1}^{10}\left(\frac{1}{2}\right)^m = \frac{\left(\frac{1}{2}\right)^{11} - \frac{1}{2}}{\frac{1}{2} - 1} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{11}}{1 - \frac{1}{2}} = 1 - \left(\tfrac{1}{2}\right)^{10} = 0.9990234375.$

4b. $\displaystyle\sum_{m=1}^{5} 3^m = \frac{3^6 - 3}{3 - 1} = 363.$

5. *Sums of squares and cubes.*

In this section, I'll use telescoping sums to find formulas for the sum of the first n squares and the first n cubes, i.e., for the sums

$$S_{n,2} = \sum_{k=1}^{n} k^2 \quad \text{and} \quad S_{n,3} = \sum_{k=1}^{n} k^3.$$

To get an idea of how this is going to work, I will first re-derive formula (3.2) in a different way. To begin, recall formula (4.3),

$$\sum_{k=1}^{n}(2k+1) = n^2 + 2n,$$

[¶]How do we evaluate the sum on the left-hand side of (4.4) if $q = 1$?

which was derived using a telescoping sum. Next, we use rules **R1.** and **R3.** to simplify the sum on the left, and express it terms of $S_{n,1}$ and n:

$$\sum_{k=1}^{n}(2k+1) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 2S_{n,1} + n. \tag{5.1}$$

Finally, comparing the right-hand side of (5.1) to the right-hand side of (4.3), we see that

$$2S_{n,1} + n = n^2 + 2n \qquad \Longrightarrow \qquad 2S_{n,1} = n^2 + n = n(n+1),$$

and dividing by 2 yields (3.2).

Let's list the steps we used in the second derivation of (3.2). To evaluate $S_1 = \sum_{k=1}^{n} k$,

i. Use the algebraic identity

$$(k+1)^2 - k^2 = 2k + 1$$

to transform the sum $\sum_{k=1}^{n}(2k+1)$ into a telescoping sum.

ii. Use the telescoping sum to evaluate the original sum:

$$\sum_{k=1}^{n}(2k+1) = \sum_{k=1}^{n}\left((k+1)^2 - k^2\right) = (n+1)^2 - 1 = n^2 + 2n.$$

iii. Use **R1.** and **R2.** to express the sum $\sum_{k=1}^{n}(2k+1)$ in terms of S_1 and other known quantities, and rearrange the terms to derive a formula for S_1.

To extend this idea to the sums $S_{n,2}$ and $S_{n,3}$, we need identities analogous to (4.2) for higher powers. These identities are

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1 \tag{5.2}$$

and

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1, \tag{5.3}$$

which *you should check* by simplifying the left-hand sides to verify that the right-hand sides are correct.

Next, we observe that the identities above imply that

$$\sum_{k=1}^{n}(3k^2 + 3k + 1) = \sum_{k=1}^{n}\left((k+1)^3 - k^3\right) = (n+1)^3 - 1 = n^3 + 3n^2 + 3n, \tag{5.4}$$

and

$$\sum_{k=1}^{n}(4k^3+6k^2+4k+1) = \sum_{k=1}^{n}\left((k+1)^4 - k^4\right) = (n+1)^4 - 1 = n^4 + 4n^3 + 6n^2 + 4n. \tag{5.5}$$

These equations follow directly from the fact that the second sum in each equation is a simple telescoping sum, (another fact that you should check!).

Finally, we use **R1.** and **R3.** to rearrange the sums on the left-hand sides of equations (5.4) and (5) to express them in terms of $S_{n,2}$ and $S_{n,3}$ (and other, known quantities) respectively. We can't do this step simultaneously, since we need to use the formula for $S_{n,2}$ to compute the formula for $S_{n,3}$, so we start with $S_{n,2}$.[‖]

It follows from the basic rules that

$$\sum_{k=1}^{n}(3k^2+3k+1) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3S_{n,2} + 3S_{n,1} + n.$$

Comparing this to the right-hand side of (5.4), shows that $3S_{n,2} + 3S_{n,1} + n = n^3 + 3n^2 + 3n$, so

$$\begin{aligned} 3S_{n,2} &= n^3 + 3n^2 + 3n - (3S_{n,1} + n) \\ &= n^3 + 3n^2 + 2n - \frac{3n(n+1)}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2}. \end{aligned}$$

Dividing this equation by 3 gives the formula for $S_{n,2}$:

$$S_{n,2} = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}. \tag{5.6}$$

The steps for evaluating $S_{n,3}$ are the same. We use the basic rules to show that

$$\sum_{k=1}^{n} 4k^3 + 6k^2 + 4k + 1 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 4S_{n,3} + 6S_{n,2} + 4S_{n,1} + n.$$

Then, we compare the right-hand side of this equation to the right-hand side of (5), which shows that

$$4S_{n,3} + 6S_{n,2} + 4S_{n,1} + n = n^4 + 4n^3 + 6n^2 + 4n.$$

And finally, we solve this equation for $S_{n,3}$, using the known formulas for $S_{n,2}$ and $S_{n,1}$:

$$\begin{aligned} 4S_{n,3} &= n^4 + 4n^3 + 6n^2 + 4n - (6S_{n,2} + 4S_{n,1} + n) \\ &= n^4 + 4n^3 + 6n^2 + 3n - (2n^3 + 3n^2 + n) - 2n(n+1) \\ &= n^4 + 2n^3 + n^2, \end{aligned}$$

and dividing by 4 gives the formula for $S_{n,3}$:

$$S_{n,3} = \sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4}. \tag{5.7}$$

[‖]Just as we needed the sum $S_{n,0} = \sum_{k=1}^{n} k^0 = n$ to evaluate $S_{n,1}$, and just as we need $S_{n,1}$ and $S_{n,0}$ to evaluate $S_{n,2}$.

Comment: In principal, we can continue in this way to find formulas for any sum of the form

$$S_{n,m} = \sum_{k=1}^{n} k^m.$$

Needless to say the formulas become more complicated as the power m increases. On the other hand, all of these formulas do share one feature. Namely, for any positive integer m, we have

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + c_m n^m + c_{n-1} n^{m-1} + \cdots + c_1 n, \tag{5.8}$$

where $c_1, \ldots, c_m$ are constant coefficients. In some applications, it is important to know the values of all the coefficients, but for the purposes of computing definite integrals, only the leading coefficient, $c_{m+1} = \frac{1}{m+1}$, is important.

Exercises.

1. Compute the sums:

a. $\displaystyle\sum_{k=1}^{10} k^2 + 3k + 2 =$

b. $\displaystyle\sum_{j=1}^{8} j^3 - 4j^2 + 7 =$

c. $\displaystyle\sum_{m=1}^{20} \left(\frac{2}{3}\right)^m =$

2. Compute the sums:

a. $\displaystyle\sum_{k=1}^{15} 2k^3 - k^2 + 3k - 1 =$

b. $\displaystyle\sum_{m=2}^{25} 3^m =$

3. Compute the following sums. Express your answers in terms of n and simplify the expressions that you find.

a. $\displaystyle\sum_{k=1}^{n} k^3 + 2k^2 + 3k + 4 =$

b. $\displaystyle\sum_{m=1}^{n} \left(\frac{2}{3}\right)^m =$

4. Consider the sum $\displaystyle\frac{1}{n}\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2$.

a. Evaluate this sum. Express your answer in terms of n.

b. Compute the limit: $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2$.

5. Verify that the identity (5.2) is correct.

6. Verify that the identity (5.3) is correct.

7. Show that (when $q \neq 1$)

$$\sum_{k=1}^{n} kq^k = \frac{nq^{n+1}}{q-1} - \frac{q^{n+1}-q}{(q-1)^2}. \tag{5.9}$$

Hint: Multiply the sum by $(q-1)$ and collect like powers of q. The result is not a telescoping sum, but it is fairly easy to evaluate using known formulas.

8. Use (5.9) to compute the sum $\sum_{k=1}^{20} \frac{k}{2^k}$.

9. a. Compute the sum $\sum_{k=1}^{n} \frac{k}{n^2} e^{-k/n}$. Express your answer in terms of n.

Hint: $e^{-k/n} = \left(e^{-1/n}\right)^k$, now use (5.9).

b. Evaluate the sum in part **a.** for $n = 10$, $n = 20$ and $n = 100$. Do you think that $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k}{n^2} e^{-k/n}\right)$ exists? Can you compute it (or guess what it is)?